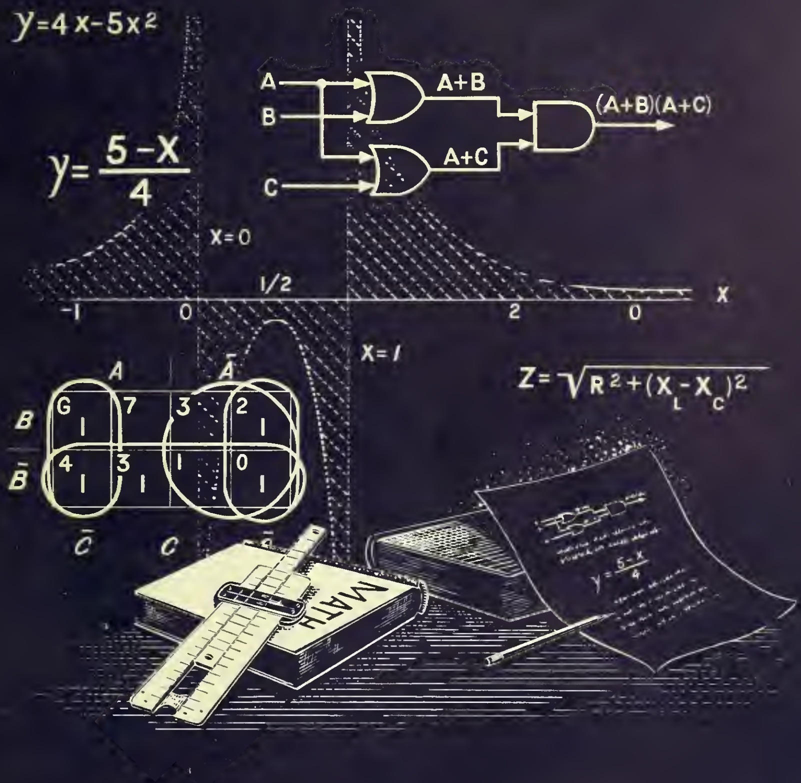

MATHEMATICS, VOL. 3

BUREAU OF NAVAL PERSONNEL

NAVY TRAINING COURSE NAVPERS 10073

PREFACE

Mathematics, Vol. 3 is the third and most advanced of a series of Navy Training Courses in mathematics. It carries forward the basic training of *Mathematics*, Vol. 1 and *Mathematics*, Vol. 2 to include the fundamentals of differential and integral calculus with an introduction to differential equations. In addition it includes subjects pertinent to digital computers such as number systems and Boolean algebra.

This training course is intended for use by operators and technicians requiring a knowledge of computer mathematics in order to understand the operation of and perform maintenance on equipment, and other personnel who need an introduction to calculus and number systems.

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THE UNITED STATES NAVY

GUARDIAN OF OUR COUNTRY

The United States Navy is responsible for maintaining control of the sea and is a ready force on watch at home and overseas, capable of strong action to preserve the peace or of instant offensive action to win in war.

It is upon the maintenance of this control that our country's glorious future depends; the United States Navy exists to make it so.

WE SERVE WITH HONOR

Tradition, valor, and victory are the Navy's heritage from the past. To these may be added dedication, discipline, and vigilance as the watchwords of the present and the future.

At home or on distant stations we serve with pride, confident in the respect of our country, our shipmates, and our families.

Our responsibilities sober us; our adversities strengthen us.

Service to God and Country is our special privilege. We serve with honor.

THE FUTURE OF THE NAVY

The Navy will always employ new weapons, new techniques, and greater power to protect and defend the United States on the sea, under the sea, and in the air.

Now and in the future, control of the sea gives the United States her greatest advantage for the maintenance of peace and for victory in war.

Mobility, surprise, dispersal, and offensive power are the keynotes of the new Navy. The roots of the Navy lie in a strong belief in the future, in continued dedication to our tasks, and in reflection on our heritage from the past.

Never have our opportunities and our responsibilities been greater.

CONTENTS

CHAPTER 1

TOPICS FOR INTRODUCTION

1-1. Introduction

The objective of *Mathematics*, Vol. 3 is to present a broad overview of advanced mathematics including number systems, Boolean algebra, calculus, and differential equations. This training course is designed to be used by both officers and enlisted men in the U.S. Navy and Naval Reserve whose duties and interests require a knowledge of this type of advanced mathematics.

The Navy is relying more heavily on mathematics today to understand and solve its problems than ever before in history. Navy computers are relied upon to solve complex problems rapidly and accurately. Operators and technicians must program and service these computers. To qualify, they must have a knowledge of the mathematics used by these computers. These are divided into two general types; digital computers operating on principles employing Boolean algebra (far different from ordinary algebra), and analog computers operating on principles employing calculus and differential equations.

To understand the material in *Math*, Vol. 3, the material in *Math*, Vol. 1 and *Math*, Vol. 2 must first be mastered. The first chapter in *Math*, Vol. 3 lists information, to be used as reference material, that was presented in *Math*, Vol. 1 and *Math*, Vol. 2. It also introduces a few new ideas that will be essential for grasping various sections of the text. It is hoped that this varied program in the beginning chapter will give the student a feel for the broad information that is at his fingertips and create in him an interest to master it.

Chapter 2 on number systems is developed to make the reader aware of other number systems and shows the use of these systems as compared to the familiar decimal number system in performing basic operations. The ease of converting numbers expressed in one number system to their equivalents in another number system is pointed out. An example of computer applications is presented.

Since the concept of Boolean algebra is new to many people, it was thought that many paths should be provided to aid in grasping its ideas. Chapters 3 and 4 provide these paths by discussing the Venn diagram, truth tables, and various diagrammatic sketches. Applications and simplification techniques are also presented.

Differentiation is then discussed in chapters 5 and 6. First, chapter 5 discusses the differentiation of algebraic functions starting from the foundation provided in *Math*, Vol. 2. Chapter 6 emphasizes differentiation of transcendental functions and their applications.

After reading and understanding the material in chapters 5 and 6 on differentiation, the student is ready to understand integration and its applications. Chapters 7 and 8 introduce integration and its application mainly to electrical examples. Integration by approximate methods is also investigated.

Series development is contained in chapter 9. The ability to represent a function by a series is emphasized and examples are included.

The main idea of chapter 10 is to help the student understand the usefulness of the differential equation in describing various simple physical systems. Mechanized solutions for a few standard forms are provided and a short discussion of analog methods is included in the last chapter.

1-2. List of Useful Formulas for Ready Reference

1. Binomial Expansions
   a. $(a \pm b)^2 = a^2 \pm 2ab + b^2$
   b. $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$
   c. $(a \pm b)^n = a^n \pm \frac{n}{1} a^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 \pm \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \ldots\ldots$

2. Polynomial Expansion

$$(a+b+c+d+\ .\ .\ .\ .\ .\ )^2=a^2+b^2+c^2+d^2 + \ .\ .\ .\ .\ . + 2a(b+c+d+\ .\ .\ .) + 2b(c+d+\ .\ .\ .) + 2c(d+\ .\ .\ .\ ) + \ .\ .\ .\ .\ .$$

3. Factoring

a. $a^2+b^2=(a+b\sqrt{-1})(a-b\sqrt{-1})$

b. $a^2-b^2=(a+b)\ (a-b)$

c. $a^3-b^3=(a-b)(a^2+ab+b^2)$

d. $a^3+b^3=(a+b)(a^2-ab+b^2)$

e. $a^4+b^4=(a^2+ab\sqrt{2}+b^2)(a^2-ab\sqrt{2}+b^2)$

f. $a^{2n}-b^{2n}=(a^n+b^n)(a^n-b^n)$

g. $a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2 + \ .\ .\ .\ .\ . + b^{n-1})$

h. $a^n-b^n=(a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2 - \ .\ .\ .\ .\ . - b^{n-1})$

if n is even.

i. $a^n+b^n=(a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2 - \ .\ .\ .\ .\ . + b^{n-1}$

if n is odd

4. The Quadratic Equation

If

$$ax^2+bx+c=0$$

then

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

or

$$x=\frac{2c}{-b\mp\sqrt{b^2-4ac}}$$

Also, if

$b^2-4ac>0$ Roots are real and unequal

$=0$ Roots are real and equal

<0 Roots are imaginary

5. Trigonometric Functions of an Angle (fig. 1–1)

a. sine (sin) $\theta=\frac{y}{z}$

b. cosine (cos) $\theta=\frac{x}{z}$

c. tangent (tan) $\theta=\frac{y}{x}$

d. cotangent (cot) $\theta=\frac{x}{y}$

e. secant (sec) $\theta=\frac{z}{x}$

f. cosecant (csc) $\theta=\frac{z}{y}$

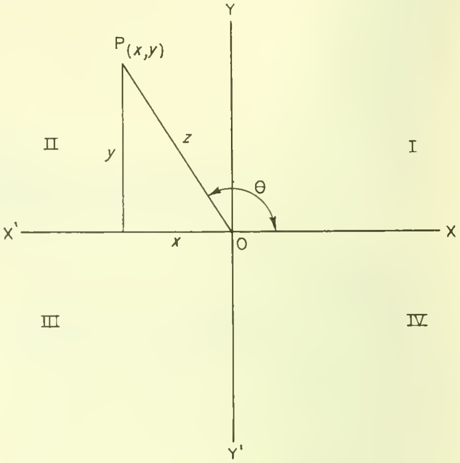


Figure 1–1.—Trigonometric functions of an angle and location of point P.

6. Sign of Trigonometric Functions (Fig. 1–1)

Quadrant	Sin	Cos	Tan	Cot	Sec	Csc
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

7. Trigonometric Functions of 0°, 30°, 45°, 60°, 90°, 180°, 270°, 360°

	0°	30°	45°	60°	90°	180°	270°	360°
Sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	1	0	1
Tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0	∞	0
Cot	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	∞	0	∞
Sec	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	∞	−1	∞	1
Csc	∞	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	∞	−1	∞

8. Fundamental Relationships Among Trigonometric Functions

a. $\sin\theta = \dfrac{1}{\csc\theta}$

b. $\cos\theta = \dfrac{1}{\sec\theta}$

c. $\tan\theta = \dfrac{1}{\cot\theta} = \dfrac{\sin\theta}{\cos\theta}$

d. $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

e. $\sec\theta = \dfrac{1}{\cos\theta}$

f. $\csc\theta = \dfrac{1}{\sin\theta}$

g. $\sin^2\theta + \cos^2\theta = 1$

h. $\sec^2\theta - \tan^2\theta = 1$

i. $\csc^2\theta - \cot^2\theta = 1$

9. Trigonometric Functions of Half Angles

a. $\sin\dfrac{\theta}{2} = \sqrt{\dfrac{1-\cos\theta}{2}}$

b. $\cos\dfrac{\theta}{2} = \sqrt{\dfrac{1+\cos\theta}{2}}$

c. $\tan\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{\sin\theta}$

$= \dfrac{\sin\theta}{1+\cos\theta}$

$= \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

10. Trigonometric Functions of Multiple Angles

a. $\sin n\theta = 2\sin(n-1)\theta\cos\theta - \sin(n-2)\theta$

b. $\cos n\theta = 2\cos(n-1)\theta\cos\theta - \cos(n-2)\theta$

11. Powers of Trigonometric Functions

a. $\sin^2\theta = \dfrac{1}{2}(1-\cos 2\theta)$

b. $\cos^2\theta = \dfrac{1}{2}(1+\cos 2\theta)$

c. $\sin^3\theta = \dfrac{1}{4}(3\sin\theta - \sin 3\theta)$

d. $\cos^3\theta = \dfrac{1}{4}(\cos 3\theta + 3\cos\theta)$

12. Trigonometric Functions of Sum or Difference of Two Angles.

a. $\sin(\theta\pm\beta) = \sin\theta\cos\beta \pm \cos\theta\sin\beta$

b. $\cos(\theta\pm\beta) = \cos\theta\cos\beta \mp \sin\theta\sin\beta$

c. $\tan(\theta\pm\beta) = \dfrac{\tan\theta\pm\tan\beta}{1\mp\tan\theta\tan\beta}$

13. Sums, Differences, and Products of Two Trigonometric Functions.

a. $\sin\theta \pm \sin\beta = 2\sin\dfrac{1}{2}(\theta\pm\beta)\cos\dfrac{1}{2}(\theta\mp\beta)$

b. $\cos\theta - \cos\beta = -2\sin\dfrac{1}{2}(\theta+\beta)\sin\dfrac{1}{2}(\theta-\beta)$

c. $\cos\theta + \cos\beta = 2\cos\dfrac{1}{2}(\theta+\beta)\cos\dfrac{1}{2}(\theta-\beta)$

d. $\tan\theta\pm\tan\beta = \dfrac{\sin(\theta\pm\beta)}{\cos\theta\cos\beta}$

e. $\sin\theta\sin\beta = \dfrac{1}{2}\cos(\theta-\beta) - \dfrac{1}{2}\cos(\theta+\beta)$

f. $\cos\theta\cos\beta = \dfrac{1}{2}\cos(\theta-\beta) + \dfrac{1}{2}\cos(\theta+\beta)$

g. $\sin\theta\cos\beta = \dfrac{1}{2}\sin(\theta+\beta) + \dfrac{1}{2}\sin(\theta-\beta)$

14. Logarithmic Expressions

If $a^x = N$ then $\log_a N = x$ which means that the logarithm of N to the base a is x.

a. $\log_a a = 1$

b. $\log_a 1 = 0$

c. $\log_a MN = \log_a M + \log_a N$

d. $\log_a \dfrac{M}{N} = \log_a M - \log_a N$

e. $\log_a M^u = u\log_a M$

f. $\log_a \sqrt[v]{M} = \dfrac{1}{v}\log_a M$

g. $\log_a a^M = M$

h. $a^{\log_a M} = M$

i. $\ln M = \log_e M$ where $e = 2.7183$ and $\ln$ is called "natural logarithm."

15. Analytic Geometry

In figure 1–1, point $P(x, y)$ is located a distance x along the X axis (abscissa) and a distance y along the Y axis (ordinate).

a. Equations of straight lines

1. $Ax+By+C=O$

slope of line $=\frac{-A}{B}$

intercept on Y axis $=\frac{-C}{B}$

2. General equation (slope-intercept form) $y=mx+b$ where $m=$ slope of line and $b=$ intercept on Y axis.

b. Equation of a circle

If the center of a circle is located at $C(a, b)$ and the radius of the circle is r, then the equation for the circle is $(x-a)^2+(y-b)^2=r^2$

16. Coordinate Transformations

When representing a point in reference to rectangular coordinates we write $P(x, y)$ as in figure 1-1. But when representing the point in reference to polar coordinates, we write $P(z, \theta)$ where z and θ are shown on figure 1-1. By examining figure 1-1, we see that:

a. $x=z \cos \theta$, and $y=z \sin \theta$

b. $z=\sqrt{x^2+y^2}$, and $\tan \theta=\frac{y}{x}$

17. Theorems on Limits

Theorem 1.

The limit of an algebraic sum, of a product, or of a quotient is equal, respectively, to the same algebraic sum, product, or quotient of the respective limits, provided, in the last named, that the limit of the denominator is not zero.

Theorem 2.

The limit of any indicated real root of a function of x is equal to the same real root of the limit of that function provided that the latter limit is positive.

$$\lim_{x \to a} \sqrt{u}=\sqrt{A} \text{ (assume } u=f(x))$$

Theorem 3.

The limit of the sin or cosine of a function of x is equal to the sine or cosine of the limit of that function.

$$\lim_{x \to a} \sin u=\sin A$$

Theorem 4.

The limit of the logarithm of a function of x is the logarithm of the limit of that function, provided that limit is positive.

$$\lim_{x \to a} \log u=\log A$$

Theorem 5.

If two functions of x approach a common limit and a third function of x is never greater than one or less than the other function for the same value of x, then the third function approaches the same limit as the other two. (See chapter 5 for further discussion.)

1-3. Inequalities

The mathematical expressions $a<b$ (a is less than b) and $c>d$ (c is greater than d), when a, b, c, and d are real numbers, are called inequalities. The first expression means that $a-b$ is a negative number and the second expression means that $c-d$ is a positive number. Two inequalities are said to be LIKE IN SENSE if their symbols are pointing in the same direction like $a>b$ and $c>d$. If their symbols are pointing in the opposite directions, they are said to be DIFFERENT IN SENSE.

The inequality $a \leqq b$ is read "a is less than or equal to b" and the continued inequality $a<b<c$ means that $a<b$ and $b<c$, and is read "a is less than b which is less than c" or "b lies between a and c".

Following are three properties of inequalities.

1. Both sides of an inequality may be increased or decreased by the same number and not change the sense of the inequality.

$$\begin{array}{rr} 5< & 8 \\ +2 & +2 \\ \hline 7< & 10 \end{array} \quad , \quad \begin{array}{rr} 5< & 8 \\ -2 & -2 \\ \hline 3< & 6 \end{array}$$

2. Both sides of an inequality may be multiplied or divided by the same positive number and not change the sense of the inequality.

$$\begin{array}{rr} 2< & 8 \\ \times 2 & \times 2 \\ \hline 4< & 16 \end{array} \quad , \quad \begin{array}{rr} 2< & 8 \\ \div 2 & \div 2 \\ \hline 1< & 4 \end{array}$$

3. If both sides if an inequality are multiplied or divided by the same negative number, the sense of the resulting inequality is reversed.

$$2<8$$

$2(-1)<8(-1)$ is not true for $-2>-8$ which shows a reverse in sense.

1-4. Summation Notation

When given a series of numbers such as:

$$a_1 + a_2 + a_3 + a_4 \ldots + a_n + \ldots$$

partial sums are often desired. That is:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$S_n = a_1 + a_2 + a_3 \ldots + a_n$$

We may also use a different notation to signify the above partial sums. The nth partial sum (S_n) may be written as follows:

$$S_n = \sum_{k=1}^{n} a_k$$

This may be read as "the partial sum S_n is equal to the summation of all terms a, from the first term where a_k is equal to a_1 and ($k=1$) to the last term where a_k is equal to a_n and ($k=n$)." The 3^{rd} partial sum of the above series may thus be written as follows:

$$S_3 = \sum_{k=1}^{3} a_k$$

1-5. Combinations and Permutations

A group of elements without reference to the order of the elements within the group is called a COMBINATION. Therefore, *ABC*, *ACB*, *BAC*, *BCA*, *CBA*, and *CAB* are made up of the same elements and are thus the same combination. The group may contain n different things (elements) and may be formed into a number of combinations (C) by taking the n different things r at a time.

Thus, the combinations formed by taking the elements within the group *ABC* two at a time are *AB*, *AC*, and *BC*.

The formula for $C(n, r)$ is as follows:

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

The notation $n!$ is used to simplify the above expression and is called factorial n. Letting n be a positive whole number, factorial n, given the symbol $n!$, is the product of all the whole numbers from 1 to n including n. Thus,

$$1! = 1$$
$$2! = 2$$
$$3! = 6$$
$$4! = 24$$
$$5! = 120$$

When finding the total number of combinations (C_T) of n things taken $1, 2, 3, 4, \ldots\ldots, n$ at a time, the following formula is used:

$$C_T = 2^n - 1$$

Each different arrangement of all or some of the elements within a group is called a PERMUTATION. Thus the permutations of the letters *ABC* taken all at a time are: *ABC*, *ACB*, *BAC*, *BCA*, *CAB*, and *CBA*. The number of permutations of n different things taken all at a time is given by the following formula:

$$P(n, r=n) = n!$$

(r is equal to the number of things taken at one time.) The number of permutations of n different things taken r at a time is:

$$P(n, r) = \frac{n!}{(n-r)!}$$

Let us consider the number of permutations of the letters in the word "SEE." Notice that "E" appears twice and there are three possible permutations. The number of permutations of n things taken all at a time, when n_1 are alike, n_2 are alike, etc., is given by the following formula:

$$P(n, n_k) = \frac{n!}{n_1! n_2! n_3! \ldots\ldots}$$

Example 1-1.

Working shifts are to be formed from six men. Each shift must consist of two men. How many different shifts may be made?

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

$$n = 6, r = 2$$

thus,

$$C(6, 2) = \frac{6!}{2!(6-2)!}$$

$$= 15 \text{ different shifts}$$

Example 1-2.

How many numbers consisting of four distinct digits may be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9?

$$P(n, r)=\frac{n!}{(n-r)!}$$

$$n=10, \text{ and } r=4$$

$$P(10, 4)=\frac{10!}{(10-4)!}$$

$$=5040 \text{ numbers}$$

Example 1–3.

What is the number of permutations in the word "REEF"?

$$P(n, n_k)=\frac{n!}{n_1!n_2!n_3!\ldots\ldots}$$

$n=4$ letters in the word "REEF", and

$n_1=2$ for the letter "E". Thus,

$$P(4, 2)=\frac{4!}{2!}=12$$

Exercise 1–1.

A. How many different sums of money can be made with a penny, a nickel, a dime, a quarter, a half dollar, and a dollar?

B. If there are 24 ships in a convoy, how many groups of three different ships may be formed?

C. If there were four ships in a screen, what is the total number of possible collisions if the screen axis is rotated 180°?

D. What is the number of permutations in the word "MISSILE".

1–6. Determinants

Let us consider the following simultaneous linear equations:

$$a_1x+b_1y=c_1$$
$$a_2x+b_2y=c_2$$

In order to solve for x multiply the first equation by b_2 and the last equation by $-b_1$. Then add the two equations to obtain the following equation:

$$(a_1b_2-a_2b_1)x=(b_2c_1-b_1c_2)$$

and then simplifying for x,

$$x=\frac{(b_2c_1-b_1c_2)}{(a_1b_2-a_2b_1)}$$

We may find the solution for y by multiplying the first equation by $-a_2$ and the last by a_1 and then adding the two equations. We then arrive at the following equation after simplifying the result of the above procedure:

$$y=\frac{(a_1c_2-a_2c_1)}{(a_1b_2-a_2b_1)}$$

Note that the denominators are alike for both x and y. The denominator may be represented by the following symbol:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

This symbol is called a DETERMINANT. The letters a_1, b_1, a_2, b_2 are called ELEMENTS and the numbers a_1 and b_2 form the PRINCIPAL DIAGONAL. The above determinant is considered a second-order determinant for it is derived from two simultaneous linear equations containing two unknowns. The relationship which is represented by the determinant is as follows:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1b_2-a_2b_1 \tag{1}$$

We now may write the solutions (x and y) of the previous simultaneous equations as:

$$x=\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$y=\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Note that the numerator and denominator are alike except for the column of coefficients of the unknown that is being solved for.

These coefficients are c_1 and c_2 in the numerator.

Example 1–4.

Solve the following simultaneous equations by determinants.

$$5y-4x=11$$
$$7y+3x=24$$

Putting into the form of determinants we have:

$$y=\frac{\begin{vmatrix}+11 & -4\\ +24 & +3\end{vmatrix}}{\begin{vmatrix}+5 & -4\\ +7 & +3\end{vmatrix}}$$

$$x=\frac{\begin{vmatrix}+5 & +11\\ +7 & +24\end{vmatrix}}{\begin{vmatrix}+5 & -4\\ +7 & +3\end{vmatrix}}$$

Expanding the determinants by equation (1) we have:

$$y=\frac{(+11)(+3)-(-4)(+24)}{(+5)(+3)-(-4)(+7)}=\frac{129}{43}$$

$$x=\frac{(+5)(+24)-(+11)(+7)}{(+5)(+3)-(-4)(+7)}=\frac{43}{43}$$

Thus, $y=3$ and $x=1$

In order to solve the more difficult determinants of the third-order, fourth-order, etc., the following rule for expanding determinants must be used to simplify the procedure.

Start with the element in the upper left-hand corner and multiply it by the determinant which is formed by striking out the element's row and column. Subtract from this quantity the next element in the row multiplied by the determinant formed by striking out the element's row and column. Add to this quantity the next element in the row multiplied by the determinant formed by striking out the element's row and column. This process is continued until the end of the row is reached.

Example 1-5.

Expand the following determinant.

$$\begin{vmatrix}1 & 2 & 1\\ 4 & 1 & 6\\ 2 & 3 & 1\end{vmatrix}$$

Using the rule for expanding determinants we have:

$$\begin{vmatrix}1 & 2 & 1\\ 4 & 1 & 6\\ 2 & 3 & 1\end{vmatrix}=1\begin{vmatrix}1 & 6\\ 3 & 1\end{vmatrix}-2\begin{vmatrix}4 & 6\\ 2 & 1\end{vmatrix}+1\begin{vmatrix}4 & 1\\ 2 & 3\end{vmatrix}$$

$$=1(1-18)-2(4-12)+1(12-2)$$
$$=-17+16+10$$
$$=9$$

Exercise 1-2.

Solve the following simultaneous equations by determinants.

a. $x+y=24$
   $x-y=12$

b. $4x+3y=7$
   $5x-7y=3$

c. $2x+7y=3$
   $3x-5y=51$

d. $2x+3y+3z=5$
   $x+y+z=0$
   $3x+2y+z=-7$

CHAPTER 2

NUMBER SYSTEMS

2–1. Introduction

All number systems are related to each other by means of digits. These are symbols, in certain positions, which represent a number. The place occupied by these digits in a positional notation in the number system determines how large or how small the number is in value. Thus, the decimal integers 692, 269, 926, and 629 all have different values, but contain the same digits.

Some number systems do not contain all of the same digits of another system. For example, compare the familiar decimal system which contains ten digits with the duodecimal system containing twelve digits,

decimal – 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
duodecimal – 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *t*, *e*

(we shall employ the symbols *t* and *e* as digits to denote ten and eleven of the decimal number system)

The simplest system is the binary system which uses only two digits, 0 and 1. The digits in this system can represent "on" or "off," "yes" or "no," and even the decimal numbers such as 962, 538, and 714 using only the two digits in a meaningful sequence.

The decimal system will be used in this chapter as a basis for discussion of the other number systems. It is used since most of our measurements are based upon the decimal system and it is the most common to our experience.

The number of different digits used in a number system is called the base or RADIX. Thus, in the binary system the radix is 2, in the ternary system the radix is 3, and in the decimal system the radix is 10.

2–2. Positional Value and Counting

Everyone has seen a car's mileage indicator. This odometer can show some very basic rules that are common to all number systems.

As your car moves along the highway, the odometer is always moving in a cyclic action. Each digit is advanced in a certain order. Thus, in the decimal system, "advancing" the digit 0 means replacing it by the digit 1, advancing digit 1 means replacing it by digit 2, advancing a 2 means replacing it by a 3, and so on until the cyclic action begins to repeat itself. At this time the digit at its adjacent left place is advanced (fig. 2–1A, B, C, and D).

In figure 2–1, this cyclic action will continue until the odometer represents the first integer it began with, which is 0000. This means the whole counting process will start to repeat itself at this time. The reason for this action is that the indicator has a limit to how large a number it can indicate. This number is 9999 and with an addition of one or more advance, the odometer will read 0000 and start to repeat itself. The maximum amount of different numbers the odometer can represent in one cycle is called the MODULUS of the counter. In our case, 10,000 is the modulus of our odometer.

Suppose we examine this odometer more closely (fig. 2–1D). The value 6392 shown on the odometer represents $6\times10^3 + 3\times10^2 + 9\times10^1 + 2\times10^0$ which equals 6392. Thus, it is seen that each digit has a positional value. The positional value of digit 2 is 2×10^0, for digit 9 it is 9×10^1, for digit 3 it is 3×10^2, and for digit 6 it is 6×10^3.

Example 2–1.

What are the positional values of 001121.12 in the ternary system (ordinarily written as 001121.12_3)?

Column Weights

3^n......	3^5	3^4	3^3	3^2	3^1	3^0		3^{-1}	3^{-2}	3^{-3}	3^{-4}	3^{-5}	3^{-6}	.....3^{-n}
....	0	0	1	1	2	1	·	1	2	0	0	0	0	....

NOTE: The succession of column weights form the radix geometric progression.

Digit	*Positional Value*	*Decimal Value*
1	$=1\times3^3$	$=27$
1	$=1\times3^2$	$=\ 9$
2	$=2\times3^1$	$=\ 6$
1	$=1\times3^0$	$=\ 1$
. (radix point)		
1	$=1\times3^{-1}$	$=\ 0.33$
2	$=2\times3^{-2}$	$=\ 0.22$
		43.55

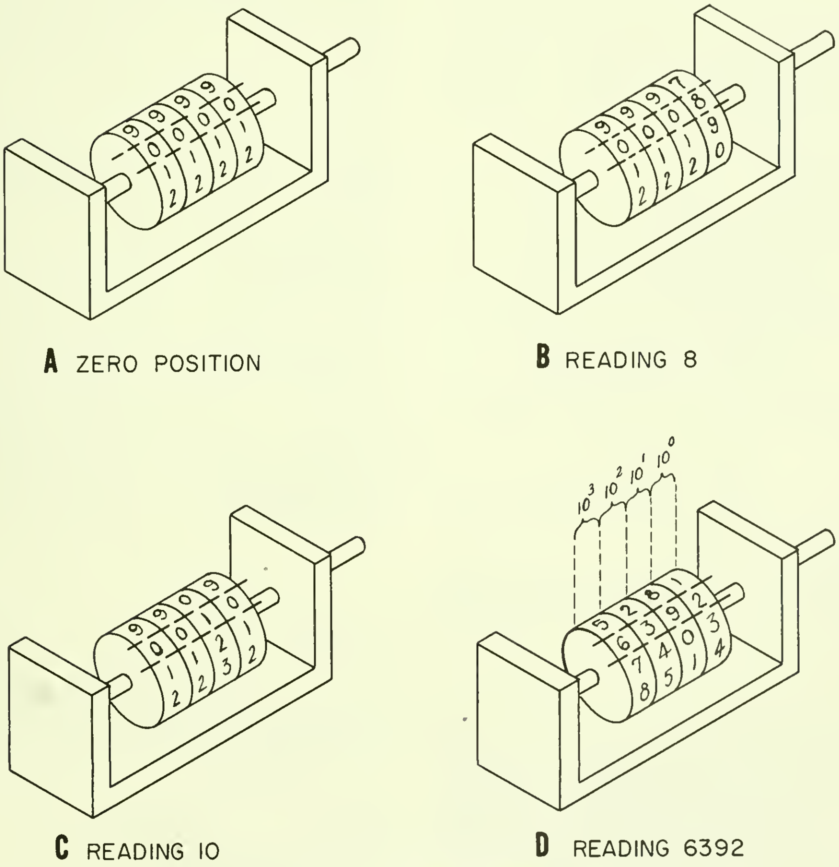


Figure 2-1.—Cyclic action of odometer.

From the above example it is seen that the point separates the column weights into two groups. Negative exponents are always on the right of the radix point (in the given example 2–1, ternary point) and positive exponents are on the left.

2–3. Converting Between Number Systems

I. *Radix r to Decimal*

It has been shown that the positions of digits is the determining factor in evaluating a number. According to the positioning conventions, the number 429.13 in the decimal system equals as follows:

$$429.13_{10} = 4(10^2) + 2(10^1) + 9(10^0) + 1(10^{-1}) + 3(10^{-2}).$$

Similarly, for the ternary (base 3), octal (base 8) and duodecimal (base 12) systems, respectively, we have the examples:

Example 2–2.

A. $20.01_3 = 2(3^1) + 0(3^0) + 0(3^{-1}) + 1(3^{-2})$

$$= 6 + \frac{1}{3^2} = 6 + \frac{1}{9} = 6.11_{10}$$

B. $036.12_8 = 0(8^2) + 3(8^1) + 6(8^0) + 1(8^{-1}) + 2(8^{-2})$

$$= 24 + 6 + \frac{8+2}{8^2} = 30.156_{10}$$

C. $0e6.et_{12} = 0(12^2) + 11(12^1) + 6(12^0) + 11(12^{-1})$

$$+ 10(12^{-2})$$

$$= 132 + 6 + \frac{66+5}{72} = 138.986\ .\ .\ ._{10}$$

where t = ten and e = eleven.

Let a, b, c, ... be digits in any number system with radix r. Then, the decimal value of *bjug.lm* is equal to:

$$bjug.lm_r = b(r^3) + j(r^2) + u(r^1) + g(r^0) + 1(r^{-1}) + m(r^{-2}).$$

Thus, to convert any given number in any system to the decimal system merely expand the number using the radix geometric progression and collect the terms. Of course the positive power (integral) part is worked out separately from the negative power (fractional) part and the two are then added together, for ease in manipulations.

II. *Decimal to Radix r*

When converting a number such as 18.375_{10} to a number with any other radix r, the integral part to the left of the decimal point must be treated separately from the fractional part located on the right of the decimal point.

There is a very handy conversion method that is used to convert any decimal integer to any other system. It is called converting by the division process. It is done by dividing the decimal integer by the new radix r and combining the remainders in reverse order to produce the new number. (Example shown after the next paragraph.)

In order to obtain the total conversion of a decimal number, we must convert the decimal fraction. This is done by a corresponding multiplication process, that is, multiply the fraction by radix r and take out the resulting integral parts in forward order to obtain the new number. Add the two parts, integral and fractional, to obtain the total conversion. Following are examples to show and familiarize the student with the preceding explanations.

Example 2–3.

Convert the decimal number 18.375 to the binary, ternary, and duodecimal systems.

A. Decimal to binary

Integral (read up)	*Fractional* (read down)
2/18	$2 \times .375 = 0.75$ ...integral part is 0.
2/9 ...remainder is 0	$2 \times .75 = 1.5$ ...integral part is 1
2/4 ...remainder is 1	$2 \times .5 = 1.0$ ...integral part is 1
2/2 ...remainder is 0	$2 \times 0 = 0.0$ ...integral part is 0
2/1 ...remainder is 0	
0 ...remainder is 1	
$\therefore 18_{10} = 10010_2$	$\therefore .375_{10} = .011_2$

$$18.375_{10} = 10010.011_2$$

B. Decimal to ternary

Integral (read up)	*Fractional* (read down)
3/18	$3 \times .375 = 1.125$...integral part is 1
3/6 ...remainder is 0	$3 \times .125 = .375$ ...integral part is 0
3/2 ...remainder is 0	$3 \times .375 = 1.125$...integral part is 1
0 ...remainder is 2	.
	.
	.
$\therefore 18_{10} = 200_3$	$\therefore .375_{10} = .101010\ .\ .\ ._3$

$$18.375_{10} = 200.101010\ .\ .\ .\ ._3$$

C. Decimal to duodecimal

Integral	*Fractional* (read down)
$12\underline{/18}$	$12 \times .375 = 4.5$...integral part is 4
$12\underline{/1}$...remainder is 6	$12 \times .50 = 6.0$...integral part is 6
0...remainder is 1	$12 \times 0 = 0$ ...integral part is 0
read up	
$\therefore 18_{10} = 16_{12}$	$\therefore .375_{10} = .460_{12}$

$$18.375_{10} = 16.460_{12}$$

III. *Radix* r_1 to r_2

We have shown how to convert a number in radix r to a decimal number and how to convert a decimal number to a number of radix r. Using these two methods, one can convert any number in radix r_1 to any number in radix r_2 by using the decimal number as a middle step.

First, convert the given number to the equivalent decimal number. Second, convert the decimal number to the desired based number. Thus, using example 2–3, to convert 200.1010_3 to base 12, expand 200.1010_3 using the radix geometric progression to find its decimal equivalent 18.375_{10} and then convert 18.375_{10} to its duodecimal equivalent by using the division process to obtain 16.460_{12}. In a forthcoming section of this chapter, a more direct method of converting radix r_1 to r_2 will be discussed.

2–4. Addition

In this section and the next two sections, the basic arithmetical operations of addition, subtraction, and multiplication will be discussed. Operations for positional number systems will be considered, and emphasis will be placed on the binary system.

Early in elementary schools, students learn and form rules for adding, subtracting, multiplying, and dividing decimal numbers. In general these rules apply to any number systems, with regard to the digit-addition and digit-multiplication tables which are unique for each system. Below are addition tables for the decimal, octal, binary, and duodecimal systems (tables 2–1, 2, 3, and 4).

TABLE 2–1.—DECIMAL ADDITION.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

TABLE 2–2.—OCTAL ADDITION.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

TABLE 2–3.—BINARY ADDITION

+	0	1
0	0	1
1	1	*10

*NOTE: This is 0 1 0 in binary digits and not ten in the decimal system.

TABLE 2–4.—DUODECIMAL ADDITION (where t = ten and e = eleven).

+	0	1	2	3	4	5	6	7	8	9	*t*	*e*
0	0	1	2	3	4	5	6	7	8	9	*t*	*e*
1	1	2	3	4	5	6	7	8	9	*t*	*e*	10
2	2	3	4	5	6	7	8	9	*t*	*e*	10	11
3	3	4	5	6	7	8	9	*t*	*e*	10	11	12
4	4	5	6	7	8	9	*t*	*e*	10	11	12	13
5	5	6	7	8	9	*t*	*e*	10	11	12	13	14
6	6	7	8	9	*t*	*e*	10	11	12	13	14	15
7	7	8	9	*t*	*e*	10	11	12	13	14	15	16
8	8	9	*t*	*e*	10	11	12	13	14	15	16	17
9	9	*t*	*e*	10	11	12	13	14	15	16	17	18
t	*t*	*e*	10	11	12	13	14	15	16	17	18	19
e	*e*	10	11	12	13	14	15	16	17	18	19	1*t*

The following examples will illustrate addition; the addition of two binaries, and then the same addition in the decimal, octal, and duodecimal systems.

Example 2–4.

101.01_2	5.25_{10}	5.2_8	5.5_{12}
1100.10_2	12.50_{10}	14.4_8	17.4_{12}
10001.11_2	17.75_{10}	21.6_8	20.9_{12}

For the binary system starting with the far right column we have $1+0=1$, so we put down 1 in the first column, and also a 1 for the next two columns. For the fourth column we have $0+0=0$, thus we put down 0 in the fourth column. For the fifth column, we have $1+1=10$, so we put down 0 for the fifth column and carry a 1. For the sixth column, we have the carried over $1+1=10$, thus we put down 0 for the sixth column and carry over 1 for the seventh column, giving the sum shown. It should be noted that the radix points must be vertically aligned just as in the familiar addition of decimals.

The binary system is used in nearly all large digital computers built to date. Most digital computers do not add more than two numbers at a time. When adding three or more numbers, the computer adds the first two numbers and then takes their sum and adds it to the third number, and so forth.

2–5. Subtraction

A. Direct Subtraction.

As in addition, it is important that the two numbers be vertically aligned on the radix point. After the numbers are properly aligned, the operation of addition or subtraction disregards the point. In a digital computer, the addition or subtraction takes place just as if all numbers were integers and information about the position of the point is kept on record and supplied when called for by the operator. It is more convenient to keep the point in each number when using pencil and paper.

Direct subtraction in any number system is performed in the same manner as in the decimal system. The following examples will illustrate subtraction.

Example 2–5.

1101.01_2	13.25_{10}	15.20_8	11.30_{12}
-0010.11_2	-02.75_{10}	-02.60_8	-02.90_{12}
1010.10_2	10.50_{10}	12.40_8	$t.60_{12}$

We will discuss "borrowing" by explaining an example and how the answer was obtained.

$$\begin{array}{r} 12_8 \\ -05_8 \\ \hline 05_8 \end{array}$$

When the thought of subtracting 5 from 2 comes into our mind, we know that 2 must be made larger in order to subtract 5 and obtain a positive answer. We must "borrow" a 1 from the digit in the left place and transfer it to the right place. To do this we must find the equivalent value of the "borrowed" 1. This equivalent value is equal to the number of ones (1's) of the radix. In this case $1+1+1+1+1+1+1+1$ for the radix is the number eight (8) and there are eight (8) ones (1's). The eight (8) ones (1's) are equal to $7+1$ in the octal system. Thus the following arithmetic can be performed in the octal system to obtain the correct answer.

"borrowed"
quantity

$$(7+1)_8+2_8-5_8=(7-5)_8+1_8+2_8$$
$$2_8+1_8+2_8=5_8.$$

B. Subtraction by Complements (No End-Around-Carry):

The following is the definition of the word, complement: That which fills up or completes, as: (a) The quantity or number required to fill a thing or make it complete. (b) That which is required to supply a deficiency; one of two mutually complementing parts.

The complement of a number is, therefore, another number which completes the original number with respect to a known reference number. The tens complement of any n-order decimal integer, p, is the difference between p and 10^n or 10^n-p. Thus 36 has 2 orders, and its tens complement is 10^2-36 or 64. By arithmetic, subtract each digit in p from 9. Then add 1 to the resulting number.

If $a-b=c$, then "a" is called the minuend, "b" is called the subtrahend, and "c" is called the difference. To subtract by tens complement, the tens complement of the subtrahend is added to the minuend. If in the result of the addition, there is one more digit than in either the minuend or subtrahend (if present the digit will always be a 1), replace the "1" by "+"; otherwise find the tens complement of the result of the addition and prefix it with a minus (−).

Example 2–6.

A. Subtract 036. from 429.

The order (number of digits) of the subtrahend and minuend must be equal before applying the complement process.

+429	minuend
+964	tens complement of subtrahend
1393	
+393	remainder

B. Subtract 361 from 263.

+263	
+639	(tens complement)
902	that is, −098 ans. (tens complement of 902)

Just like the tens complements in the decimal system, we have the two complements in the binary system. The twos complement of an n-order binary integer, k, is equal to 2^n-k. Thus 1101 has 4 orders, and its twos complement is 2^4-1101 or $10000-1101=0011$. By arithmetic, change each digit in the subtrahend and then add 1 to the resulting number. To subtract by twos complement, the twos complement of the subtrahend is added to the minuend. If there is an extra-column 1 in the sum, replace it by a "+"; otherwise find the twos complement of the result of the addition and prefix it with a "−."

Example 2–7.

A. Subtract 110 from 11001 using the twos complement method. Note the order (number of digits) of the subtrahend and minuend must be equal before applying the complement process.

+11001	minuend
+10011	twos complement of subtrahend
101100	sum
+01100	remainder

B. Subtract 1101 from 1000.

+1000	
+0011	(twos complement)
1011	that is, −0101

C. Subtraction by Complements (With End-Around-Carry).

Consider any n-order decimal integer p. Its nines complement is $(10^n-p)-1$ which is its tens complement minus one. Thus, the nines complement of 32 is 67 since its tens complement is 68. By arithmetic, subtract each digit in the decimal integer from nine to obtain the nines complement. In order to subtract by the nines complement, the nines complement of the subtrahend is added to the minuend. If there is an extra-column 1, replace it by "+" and then add 1 by the process called "end-around-carry"; otherwise nines complement the result at the addition and prefix "−."

Example 2–8.

A. Subtract 32 from 475.

+475	
+967	(nines complement)
1442	
+1	(end-around-carry to units column)
443	

B. Subtract 451 from 360.

+360	
+548	(nines complement)
908	
−091	(nines complement of addition)

Just like the nines complement in the decimal system, we have the ones complement in the binary system. The ones complement of n-order binary integer k is equal to $(2^n-k)-1$. By arithmetic, change each digit in k to the other digit to obtain the ones complement. This process is termed inversion and is fundamental to computer operations using ones complement. Thus, the ones complement of 10110 is 01001. In order to subtract by the ones complement, the ones complement of the subtrahend is added to the minuend. If there is an extra-column 1, replace it by "+" and then add 1 by the process called "end-around-carry"; otherwise, ones complement the result of the addition and prefix "−."

Example 2–9.

A. Subtract 10110 from 11000.

+11000	
+01001	(ones complement)
100001	
+1	(end-around-carry)
00010	to units column

B. Subtract 10110 from 10000.

698-990 O-64—2

$$
\begin{array}{ll}
+10000 & \\
\underline{+01001} & \text{(ones complement)} \\
11001 & \\
-00110 & \text{(ones complement of addition)}
\end{array}
$$

2-6. Multiplication

1. *Direct*

In a previous section of this chapter, we discussed digit-addition tables for number systems. Likewise, each number system has a digit-multiplication table. Students in the primary grades memorize the decimal digit-multiplication table. Following are the decimal, octal, duodecimal, and binary multiplication tables (tables 2-5, 2-6, 2-7, and 2-8).

In multiplication in the decimal system, certain rules are followed which use the decimal digit multiplication and decimal-digit addition tables. These

TABLE 2-5.—DECIMAL MULTIPLICATION.

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

TABLE 2-6.—OCTAL MULTIPLICATION.

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	10	12	14	16
3	0	3	6	11	14	17	22	25
4	0	4	10	14	20	24	30	34
5	0	5	12	17	24	31	36	43
6	0	6	14	22	30	36	44	52
7	0	7	16	25	34	43	52	61

TABLE 2-7.—DUODECIMAL MULTIPLICATION (where t = ten and e = eleven).

×	0	1	2	3	4	5	6	7	8	9	*t*	*e*
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	*t*	*e*
2	0	2	4	6	8	*t*	10	12	14	16	18	1*t*
3	0	3	6	9	10	13	16	19	20	23	26	29
4	0	4	8	10	14	18	20	24	28	30	34	38
5	0	5	*t*	13	18	21	26	2*e*	34	39	42	47
6	0	6	10	16	20	26	30	36	40	46	50	56
7	0	7	12	19	24	2*e*	36	41	48	53	5*t*	65
8	0	8	14	20	28	34	40	48	54	60	68	74
9	0	9	16	23	30	39	46	53	60	69	76	83
t	0	*t*	18	26	34	42	50	5*t*	68	76	84	92
e	0	*e*	1*t*	29	38	47	56	65	74	83	92	*t*1

TABLE 2-8.—BINARY MULTIPLICATION.

×	0	1
0	0	0
1	0	1

rules are well known and apply to direct multiplication in any number system. To illustrate this, the following examples are given.

Example 2–10.

A. Binary	B. Decimal	C. Octal
110101	53	65
×1101	×13	×15
110101	159	411
00000	53	65
110101	689	1261
110101		
1010110001		

II. *Multiplication by Over-And-Over Addition*

The over-and-over addition method of multiplication dispenses entirely with multiplication tables. This method makes use of only addition to obtain the result. An outline for this type of multiplication is presented below.

We make use of two registers. One called the accumulator which, at the start, contains 00000. The other contains the multiplier and keeps account of the number of additions that have been performed. To illustrate, the following examples are given.

Example 2–11.

A. Multiply 232 × 462.

Accumulator	*Multiplier*	*Manipulations*
000000	232	Multiplicand is recorded.
000462		
000462	231	
000462		
000924	230	First order digit is zero. Move multiplicand one place to left.
004620		
005544	220	
004620		
010164	210	
004620		
014784	200	Second order digit is zero. Move multiplicand one place to left.
046200		
060984	100	
046200		
107184	000	Answer.

B. Multiply $11010_2 \times 1011_2$.

Accumulator	*Multiplier*	*Manipulations*
000000000	11010	Move multiplicand one place left.
000010110		
000010110	11000	Move multiplicand two places left.
001011000		
001101110	10000	Move multiplicand one place left.
010110000		
100011110	00000	Answer.

2–7. Division

I. *Direct*

Direct division in any system uses the same general rules used in decimal division. To illustrate, four examples are given. Notice the use of the general rules.

Example 2–12.

Binary

```
         1101.0
1001/1110101.0
     1001
      1011
      1001
        100
        000
        1001
        1001
        0000
```

Decimal

```
  13.0
9/117.0
  9
   27
   27
   00
```

Octal

```
   15.0
11/165.0
   11
    55
    55
    00
```

Quinary (Base 5)

```
    23
14/432
   33
   102
   102
   000
```

II. *Over-and-Over Subtraction*

In multiplication, we had the over-and-over addition process. In division, we have the opposite in manipulation called, over-and-over subtraction. We still make use of two registers. One called the accumulator which, at the start, contains 00000. The other contains the quotient and keeps account of the number of subtractions that have been performed. To illustrate, the following example is given in the decimal system.

Example 2–13.

Divide 40320 by 126:

Accumulator	Quotient Register	Manipulations
40320	0000.0	Zero in Q.R.
12600	0100.0	Add 10^2 to Q.R.
27720	0100.0	Subtract in Accum.
12600	0100.0	Add 10^2 to Q.R.
15120	0200.0	Subtract in Accum.
12600	0100.0	Add 10^2 to Q.R.
02520	0300.0	Subtract in Accum.
01260	0010.0	Add 10^1 to Q.R.
01260	0310.0	Subtract in Accum.
01260	0010.0	Add 10^1 to Q.R.
00000	0320.0	Answer.

The preceding example can be explained by the following outline.

(1) Put zero into the quotient register and dividend into the accumulator.

(2) Subtract from the dividend, the largest value of the divisor times 10^n that produces a positive value.

(3) Repeat (2) again until the quotient accumulator equals zero or approximately zero. The quotient register will indicate the quotient, and the accumulator will indicate the remainder.

Again we will illustrate the above outline by use of the following example in the binary system.

Example 2–14.

Divide 1101011.01 by 1101:

Accumulator	Quotient Register	Manipulations
1101011.01	0000.00	Zero in Q.R.
1101000.00	1000.00	Add 2^3 to Q.R.
0000011.01	1000.00	Subtract in Accum.
0000011.01	0000.01	Add 2^{-2} to Q.R.
0000000.00	1000.01	Answer.

III. *Division by Reciprocal Multiplication*

The division of x by y can also be performed by multiplying x times $\frac{1}{y}$. The expression $\frac{1}{y}$ is called the reciprocal of y and can be found approximately by means of multiplication and subtraction only. To find the approximate reciprocal of a DECIMAL number y, the following equation is used.

$A_1 = A_0(2 - A_0y)$

A_0 = first approximation between 0 and $\frac{2}{y}$

A_1 = closer approximation of reciprocal.

The equation is used over and over again until A_0 equals A_1 or very nearly so. The following example illustrates the method.

Example 2–15.

What is the reciprocal of 4? Let the first approximation be 0.10.

$A_1 = A_0(2 - A_0y) = .1\,[2 - .1(4)]$

$A_1 = .16$

$A_2 = .16\,[2 - .16(4)] = .2176$

$A_3 = .2176[2 - .2176(4)] = .24580096$

$A_4 = .24580096[2 - .24580096(4)] = .250000$ (approx.)

When finding the reciprocal of a BINARY number, the equation $A_1 = A_0(10 - A_0y)$ is used when y = the binary number.

2–8. Direct Conversions of Radix r_1 to r_2

I. *Explosion Method (for whole numbers only)*

In a previous section of this chapter it was mentioned that a more direct method of converting radix r_1 to r_2 would be discussed. There are other methods in this conversion, but the so called explosion method is the most common. This method is used on whole numbers only, even though it represents the expansion of the radix geometric progression. The treatment of fractional parts is tedious and too involved for discussion here.

The explosion method in outline form is as follows:

Step (1) Perform all arithmetic operations in the desired base.

Step (2) Express the base of the original system in terms of the base of the desired system.

Step (3) Multiply the number obtained in step (2) by the leftmost digit and add the product to the next digit on the right of the original number.

(NOTE: It may be necessary to convert each digit of original number to that of desired base.)

Step (4) Repeat step (3) as many times as there are digits. The final sum is the answer.

Example 2–16.

A. Convert 2143_5 to radix 7.

Step (2) $10_5 = 5_7$

Step (3)

$$\begin{array}{llll}
(\;2_7 & 1_7 & 4 & 3_7)_5 \\
\times 5_7 & +13 & +106 & +601 \\
\hline
13_7 & 14 & 113 & 604_7 = 2143_5 \\
 & \times 5_7 & \times 5 & \text{Answer} \\
\hline
 & 106 & 601 &
\end{array}$$

B. Convert 2110_3 to radix 8

Step (2) $10_3 = 3_8$

Step (3)

$$\begin{array}{llll}
(\;2_8 & 1_8 & 1_8 & 0_8)_3 \\
\times 3_8 & \times 6_8 & +25_8 & +102_8 \\
\hline
6_8 & 7_8 & 26_8 & 102_8 = 2110_3 \\
 & \times 3_8 & \times 3_8 & \text{Answer} \\
\hline
 & 25_8 & 102_8 &
\end{array}$$

II. *Division Method (for whole numbers only)*

The division method in outline form is as follows:

Step (1) All arithmetic is performed in the original number base.

Step (2) Express the desired or new base in terms of the original system.

Step (3) Divide the original number and its successive quotients by the number found in step (2) and take the remainders in reverse order to produce the new number. (Note: Remainder digits may have to be converted to new base.)

Example 2–17.

A. Convert 2143_5 to radix 7.

Step (2) $10_7 = 12_5$

Step (3)

$$\begin{array}{r}
132 \\
12_5 \,\overline{)\,2143} \\
\underline{12} \\
44 \\
\underline{41} \\
33 \\
\underline{24} \\
4_5 = \text{1st remainder} = 4_7
\end{array}$$

$$\begin{array}{r}
11 \\
12_5 \,\overline{)\,132} \\
\underline{12} \\
12 \\
\underline{12} \\
0 = \text{2nd remainder} = 0_7
\end{array}$$

$$\begin{array}{r}
0 \\
12_5 \,\overline{)\,11_5} \\
\underline{00} \\
11_5 = \text{3rd remainder} = 6_7
\end{array}$$

$\therefore 604_7 = 2143_5$

B. Convert 2110_3 to radix 8

Step (2) $10_8 = 22_3$

Step (3)

$$\begin{array}{r}
22 \\
22_3 \,\overline{)\,2110_3} \\
\underline{121} \\
200 \\
\underline{121} \\
002_3 = \text{1st remainder} = 2_8
\end{array}$$

$$\begin{array}{r}
1 \\
22_3 \,\overline{)\,22_3} \\
\underline{22} \\
00 = \text{2nd remainder} = 0_8
\end{array}$$

$$\begin{array}{r}
0 \\
22 \,\overline{)\,1_3} \\
\underline{0} \\
1 = \text{3rd remainder} = 1_8
\end{array}$$

$\therefore 102_8 = 2110_3$

III. *Conversion of Binary to Octal*

After one has learned to recognize immediately that binary 010 is 2, 011 is 3, 100 is 4, 101 is 5, 110 is 6, and 111 is 7, it is quick and easy to convert a binary number to an octal number.

We will illustrate by converting the binary number 011,011,101 to the octal system.

Step (1) First, separate the digits into groups of three to get 011, 011, 101_2, starting from the binary point.

Step (2) Second, write down the decimal equivalent of the group to get the octal equivalent 335_8.

2–9. Examples of Computer Technique of Addition

The computer technique of addition discussed here includes the process of half add, carry, and final sum. The first logical conclusion in addition is the examination of the augend (first number to which others are added) and the addend (next number added to augend) to determine the half add process. The half add process consists of the sum of the numbers neglecting the carryover digits.

The second logical conclusion is the number of carryovers and noting their place in reference to the half add conclusion. The third logical conclusion is the final sum of the half add conclusion and the carryover conclusion. The following example will show these conclusions.

Example 2–18.

Add 3561_{10} and 2148_{10}.

```
 3561   augend
+2148   addend
 5609   half add (neglect carryovers)
+010-   carryovers
 5709   final sum (total)
```

When the addition of half add conclusion and the carryover conclusion produces more carryovers (lower level carryovers), the three conclusions are repeated until a total final sum is produced. To illustrate, the following examples are given.

Example 2–19.

Add 4965_{10} and 5247_{10}.

```
  4965   augend
 +5247   addend
  9102   half add
 +111-   carryovers
  0212   half add
+1000-   carryovers
 10212   final sum (total)
```

Example 2–20.

Add 101101_2 and 100111_2.

```
  101101   augend
 +100111   addend
  001010   half add
+100101-   carryovers
 1000000   half add
+001010-   carryovers
 1010100   final sum (total)
```

CHAPTER 3

BOOLEAN ALGEBRA

3–1. Introduction

The father of Boolean algebra was George Boole (1815–1864), who was an English logician and mathematician. In the spring of 1847, he wrote a pamphlet called *Mathematical Analysis of Logic*. But later (1854), Boole wrote a more mature statement of his logical system in a much larger work called, *An Investigation of the Laws of Thought*, in which are founded the mathematical theories of logic. He did not regard logic as a branch of mathematics, but he did point out that a close analogy between the symbols of algebra and those symbols which he devised to represent logical forms does exist.

Boolean algebra lay almost dormant until its useful application to the new field of electronic computers was discovered. Boolean algebra has now become an important subject to be learned in order to understand electronic computer circuits.

3–2. Classes and Elements.

In our universe we can logically think there are two divisions; all things of interest in any discourse are in one division, and all other things not of interest are in the other division. These two divisions are called, respectively, the universal class (I), and the null class (O).

We now ask what makes up the universal class? We know that the null class is made up of all things not under discussion. But the universal class is composed of all things, called elements, which are of interest. These elements can be grouped together to form many combinations. In Boolean logic, these combinations are called classes and should not be confused with the null class or universal class. These classes are a part of the universal class. Each class is dependent upon its elements and the possible states (stable, nonstable, or both) the elements can be in.

Boolean algebra is that algebra which is based on Boolean logic and concerned with all elements having only two possible stable states and no unstable states.

To determine the number of classes or combinations of elements in Boolean algebra, solve for the numerical value of 2^n where n equals the number of elements which are sometimes called variables. It should be noted that if there are no elements then $2^n = 2^0 = 1$. This means that when there are no elements there is one class, the null class. Also since there are no elements, the universal class does not exist.

3–3. Venn Diagram

The Venn diagram is a topographical picture of logic, composed of the universal class divided into classes depending on the n number of elements. Let us consider all submarines and all underwater sound sources that are not submarines. The Venn diagram pictures this in figure 3–1A. These two classes have no members in common, and there is no underwater sound source which is not a member of one of them. The combination of these two classes comprises the total of the universal class.

Now, let us consider the universal class as containing another class of all atomic powered sound sources. The Venn diagram of this logic is shown in figure 3–1B. By examining this diagram we see that there are four areas. Thus the universal class is divided into four classes as follows:

(1) Submarines and not atomic
(2) Submarines and atomic
(3) Atomic and not submarines
(4) Not submarines and not atomic

Figure 3–1C shows these classes separately as shaded areas with x equaling submarines and y equaling atomic powered sound sources. Using x and y we have as follows:

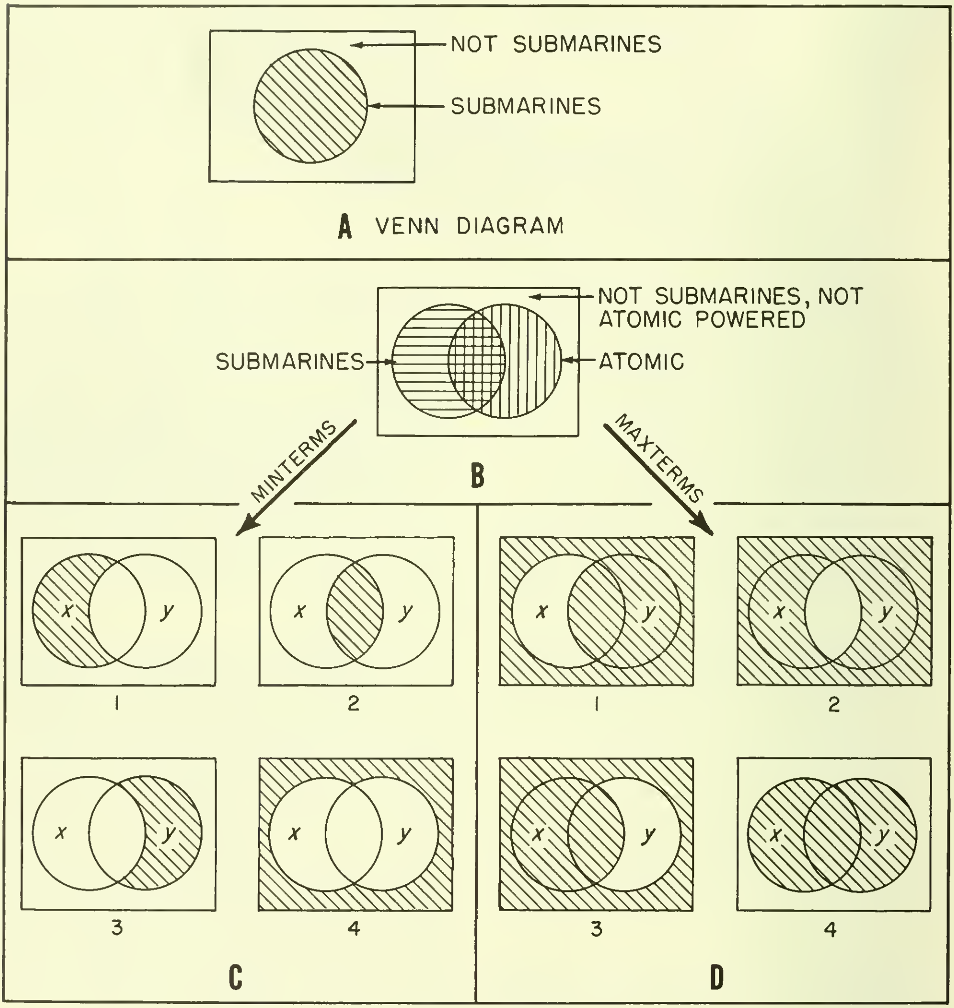


Figure 3-1.—The Venn diagram.

(1) x and not y
(2) x and y
(3) y and not x
(4) not x and not y

These four classes are called minterms. They are called minterms for they represent the four minimum classes of elements. The opposite of minterms is maxterms. These classes are represented in figure 3-1D and are written as follows:

(1) y or not x
(2) not x or not y
(3) x or not y
(4) x or y

They are called maxterms for they represent the four maximum classes of elements. For explanation, (4) x or y, contains all elements except those being both, not submarines and not atomic. In succeeding sections, we will discuss minterms and maxterms in more detail.

3–4. Basic Expressions

In the previous section, the Venn diagram was shown to represent a picture of logic. The logic shown was written in longhand, and used the important words "and," "or," and "not." These words form the basis for combining classes in Boolean algebra logic description. The symbols for these words are "·," "+," and "¯," respectively. The following is an example of this notation.

Example 3–1.

$a \cdot b$ reads a and b

$a+b$ reads a or b

$\bar{a}$ reads not a

Thus, in figure 3–1C we can write the four classes as:

(1) $x \cdot \bar{y}$ or $x(\bar{y})$
(2) $x \cdot y$ or $x(y)$
(3) $y \cdot \bar{x}$ or $y(\bar{x})$
(4) $\bar{x} \cdot \bar{y}$ or $\bar{x}(\bar{y})$

In figure 3–1D the four classes may be written in Boolean algebra notation as follows:

(1) $y+\bar{x}$
(2) $x+\bar{y}$
(3) $x+\bar{y}$
(4) $x+y$

It should be noted that the complement of x is equal to $\bar{x}$. The complement of y is equal to $\bar{y}$. The complement of a minterm is a maxterm. Their logical sum describes the universal class containing all elements.

3–5. Application to Switching Circuits

Due to the fact that Boolean algebra is based upon elements having two possible stable states, it becomes very useful in representing switching circuits. The reason for this is that a switching circuit can be in only one of two possible stable states. That is, the state of being open or the state of being closed. These two states may be 0 and 1, respectively (fig. 3–2).

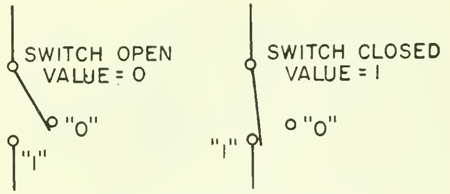


Figure 3–2.—Switch value representation.

Since the binary number system (chap. 2) consists of only the symbols 0 and 1, we employ these symbols in Boolean algebra and call this *binary Boolean algebra*. This chapter and the next will be based on the discussion of binary Boolean algebra and its switching circuit applications.

3–6. The AND Operation

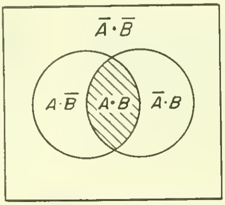
Let us consider the Venn diagram in figure 3–3A. Its classes are labeled using the basic expressions of Boolean algebra. Note there are two elements, or variables, A and B. The shaded area represents the class of elements that are $A \cdot B$ in Boolean notation and is expressed by Boolean algebra as:

$$f(A,B)=A \cdot B.$$

This expression is called an *AND* operation because it represents one of the four minterms discussed in section 3–3. *AND* indicates class intersection and both A and B must be considered together.

We can conclude then that a minterm of n variables is a logical product (the use of the symbol of pure algebraic multiplication to represent the *AND* operation) of these n variables with each variable present in either its noncomplemented or its complement form, and is considered an *AND* operation. Two variables have four minterms, AB, $\bar{A}B$, $A\bar{B}$, and $\bar{A}\bar{B}$.

In this algebra, any group of variables, which represents an expression of logic, is called a function and is symbolized f. For any Boolean function there is a corresponding truth table which shows, in tabular form, the true condition of the function for each way in which conditions can be assigned to its variables. In binary Boolean algebra, 0 and 1 are the symbols assigned to the variables of any function. See figure 3–3B for the *AND* operation

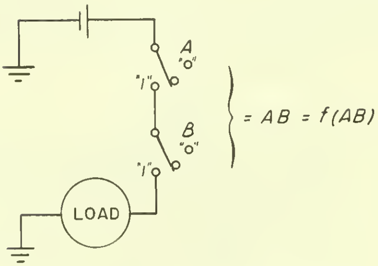


A VENN DIAGRAM

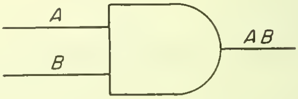
$f(A,B)=AB$

A	B	f(A,B)=AB
0	0	0
0	1	0
1	0	0
1	1	1

B TRUTH TABLE



C AND SWITCHING CIRCUIT



D LOGIC DIAGRAM MECHANIZATION OF $f(A,B)=AB$

Figure 3–3.—The *AND* operation.

function of two variables and its corresponding truth table.

The truth table example of an *AND* operation (fig. 3–3B) can be seen to be true if one thinks of the logic involved. AB is equal to A *AND* B which is the function $f(A, B)$. Thus, if either A or B takes the condition of 0, or both, then the function $f(A, B)=AB$ is equal to 0. But if not, and both A and B take the condition of 1 then the *AND* operation function has the condition of 1.

By means of switching circuits, if two or more switches are placed in series, the result is known as an *AND* circuit. Inspection of the arrangement in figure 3–3C shows that the resulting circuit will transmit only if both A and B are closed, i.e., equal to 1. If either switch A or switch B is open, i.e., equal to 0, then the resulting circuit will not transmit. This representation (fig. 3–3C) is the switching circuit for an *AND* operation.

In any digital computer equipment, there will be many circuits like the one described above. There may be other circuits sometimes called gates which act like the above circuit (fig. 3–3C) and produce the same effect. In order to analyze circuit operation, it is necessary to refer frequently to these circuits or gates without looking at their switch arrangements. This is done by logic diagram mechanization. The logic diagram for the *AND* operation is given in figure 3–3D. This means that there are two inputs, A and B, into an *AND* operation circuit producing the function in Boolean form of AB. These diagrams simplify equipment circuits by indicating operations without drawing all the circuit details.

3–7. The OR Operation

Let us now consider the Venn diagram in figure 3–4A. Note there are two elements, or variables, A and B. The shaded area represents the class of elements that are $A+B$ in Boolean notation and is expressed by Boolean algebra as:

$$f(A,B)=A+B.$$

This expression is called an *OR* operation for it represents one of the four maxterms discussed in section 3–3.

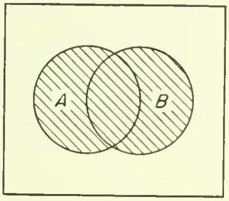
We can conclude then that a maxterm of n variables is a logical sum (the use of the symbol of pure algebraic addition to represent the *OR* operation) of these n variables where each variable is present in either its non-complemented or its complemented form. Two variables have four maxterms, $A+B$, $A+\overline{B}$, $\overline{A}+B$, and $\overline{A}+\overline{B}$.

In figure 3–4B the truth table of an OR operation is shown. The examples' truth table can be seen to be true if one thinks of the logic involved. $A+B$ is equal to *A OR B* which is the function $f(A, B)$. Thus if A or B takes the value of 1, then $f(A, B)$ must equal 1. If not, then the function equals zero.

By means of switching circuits, if two or more switches are placed in parallel, the result is known as an *OR* circuit. Inspection of the arrangement in figure 3–4C shows that the resulting circuit will transmit if either A or B is in the closed position, i.e., equal to 1. If, and only if, both A and B are open, i.e., equal to 0, then the gate will not transmit. This representation (fig. 3–4C) is the switching circuit for an *OR* operation which is sometimes referred to as an *OR* gate.

The logic diagram for the *OR* operation is given in figure 3–4D. This means that there are two inputs, A and B, into an *OR* operation circuit producing the function in Boolean form of $A+B$. Note that the diagram differs in shape from the logic diagram of an *AND* operation circuit as in figure 3–3D.

Let us now consider the expression $f(A, B) = A\overline{B}+\overline{A}B$ which reads "*A AND NOT B OR NOT A AND B*" which can be expressed as "*A* or *B* but not both." This *OR* expression differs from the *OR*

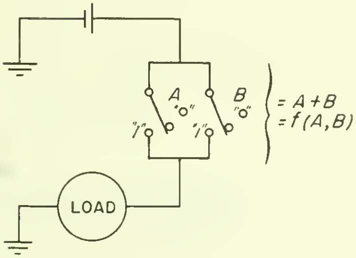


A VENN DIAGRAM

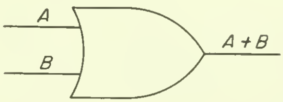
$f(A,B)=A+B$

A	B	$f(A,B)=A+B$
0	0	0
0	1	1
1	0	1
1	1	1

B TRUTH TABLE



C OR SWITCHING CIRCUIT



D LOGIC DIAGRAM MECHANIZATION OF $f(A,B)=A+B$

Figure 3–4.—The *OR* operation.

expression $f(A, B)=A+B$. $f(A, B)=A\bar{B}+\bar{A}B$ is called an "exclusive *OR*" expression, while $f(A, B) =A+B$ is called an "inclusive *OR*" expression. Their difference can best be explained by the following illustration.

The concepts *AND*, *OR*, and *NOT* (complement) are familiar ones, used by all of us in expressing ideas. We tell the waitress we would like some ice cream *AND* cake, *OR* some strawberries *AND* whipped cream. The *OR* may be either exclusive or inclusive, depending upon how hungry we are. Thus, we can see that the inclusive *OR* circuit (fig. 3-4C) can have either switch A closed, switch B closed, or both closed to transmit. As to an exclusive *OR* expression, the switching circuit is as shown in figure 3-5. Switch A or B may be closed, but not both, for the circuit to transmit.

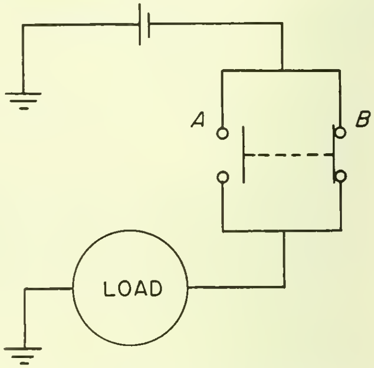
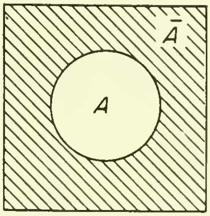


Figure 3-5.—The exclusive *OR* switching circuit.

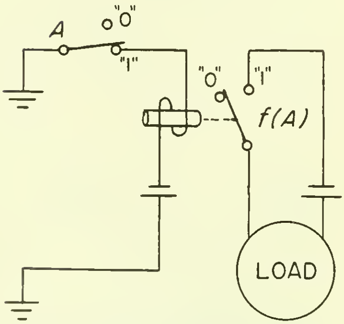


A VENN DIAGRAM

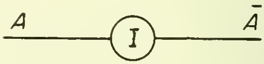
$f(A) = \bar{A}$

A	$f(A) = \bar{A}$
1	0
0	1

B TRUTH TABLE



C NOT SWITCHING CIRCUIT



D LOGIC DIAGRAM MECHANIZATION OF $f(A) = \bar{A}$

Figure 3-6.—The *NOT* operation.

3–8. The NOT Operation

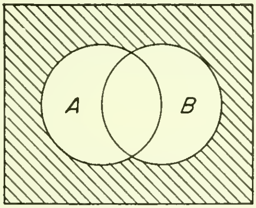
We now refer to the Venn diagram in figure 3–6A. The shaded area represents the complement of A which is noted in Boolean algebra as $\overline{A}$ and read "*NOT A*." The expression $f(A)=\overline{A}$ is called a *NOT* operation.

In figure 3–6B the truth table of a *NOT* operation is shown. It is explained by understanding the *NOT* operation circuit. The requirements of a *NOT* circuit are that a signal injected at the input produce the complement of the signal at the output, and furthermore that the complement of the signal at the input produce the signal at the output. Thus, in figure 3–6C it can be seen that when switch A is closed, i.e., equal to 1, the relay opens the circuit to the load. When switch A is open, i.e., equal to 0, the relay completes a closed circuit to the load.

The logic diagram for the *NOT* operation is given in figure 3–6D. This means that A is the input to a *NOT* operation circuit, called an inverter (used to perform the inversion process discussed in chapter 2–5, example 2–8), and gives an output of $\overline{A}$.

3–9. The NOR Operation

Figure 3–7A shows the Venn diagram for a *NOR* operation. The shaded area represents the quantity, *A OR B*, negated and is a minterm expression, as shown in figure 3–1C (4). Notice that $\overline{A+B}$ is equal to the minterm expression $\overline{A}\,\overline{B}$ for their Venn diagrams are identical. This relationship is an application of DeMorgan's Theorem which is described in section 4–2 of chapter 4 of this training course. This minterm is discussed in section 3–3 of this chapter.

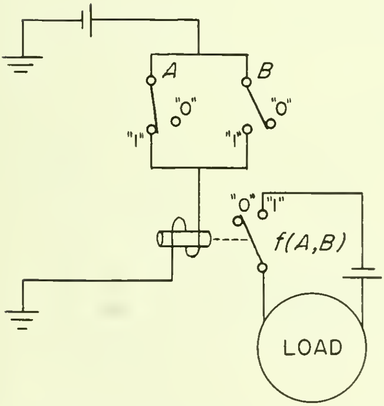


A VENN DIAGRAM

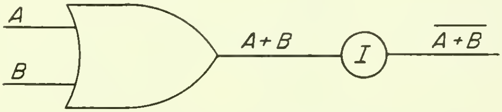
$f(A,B)=\overline{A+B}$

A	B	$A+B$	$f(A,B)=\overline{A+B}$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

B TRUTH TABLE



C NOR SWITCHING CIRCUIT



D LOGIC DIAGRAM MECHANIZATION OF $f(A,B)=\overline{A+B}$

Figure 3–7.—The *NOR* operation.

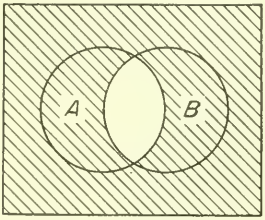
The truth table for the *NOR* operation is given in figure 3–7B. The table shows that if either A or B is equal to 1, then $f(A, B)$ is equal to 0. Furthermore, if A and B equal 0, then $f(A, B)$ equals 1.

The *NOR* operation is a combination of the *OR* operation and the *NOT* operation. The *NOR* switching circuit (fig. 3–7C) is the *OR* circuit put in series with the *NOT* circuit. If either switch A, switch B, or both are in the closed position, i.e., equal to 1, then there is no transmission to the load. If switches A and B are open, i.e., equal to 0, then current is transmitted to the load.

The logic diagram mechanization of $f(A, B) = \overline{A+B}$ (*NOR* operation) is shown in figure 3–7D. It uses both the *OR* logic diagram and the inverter (*NOT* operation) logic diagram. The *NOR* logic diagram mechanization shows that there are two inputs, A and B, into an *OR* circuit producing the function in Boolean form of $A+B$. This function is the input into the inverter which gives the output, in Boolean form, of $\overline{A+B}$. Note that the whole quantity of $A+B$ is complemented and not the separate variables.

3–10. The NAND Operation

The *NAND* operation has a Venn diagram which is shown in figure 3–8A. The shaded area represents A *AND* B, the quantity negated (*NOT*), and is a maxterm expression, as shown in figure 3–1D(2). Notice that $\overline{AB}$ is equal to the maxterm expression

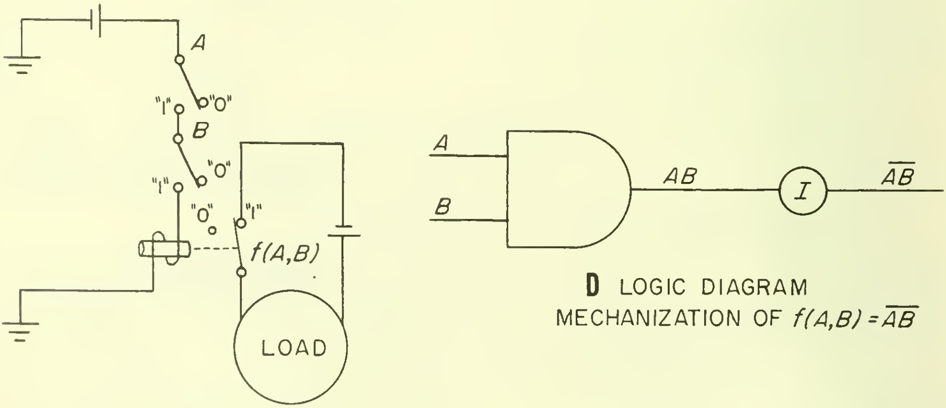


A VENN DIAGRAM

$f(A,B) = \overline{AB}$

A	B	AB	$f(A,B) = \overline{AB}$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

B TRUTH TABLE



C NAND SWITCHING CIRCUIT

D LOGIC DIAGRAM MECHANIZATION OF $f(A,B) = \overline{AB}$

Figure 3–8.—The *NAND* operation.

$\overline{A}+\overline{B}$ for their Venn diagrams are identical. This maxterm is discussed in section 3–3.

The truth table is shown for the *NAND* operation in figure 3–8B. When A and B equal 1, then $f(A, B)$ is equal to 0. In all other cases, the function is equal to 1.

The *NAND* operation is a combination of the *AND* operation and the *NOT* operation. The *NAND* switching circuit (fig. 3–8C) is the *AND* circuit put in series with the *NOT* circuit. If either switch A or B is open, i.e., equal to 0, then current is transmitted to the load. If both switch A and B are closed, i.e., equal to 1, then there is no transmission to the load.

The logic diagram mechanization of $f(A, B) = \overline{AB}$ (*NAND* operation) is shown in figure 3–8D. The *AND* operation logic diagram and the *NOT* operation logic diagram are both used. The *NAND* logic diagram mechanization shows that there are two inputs, A and B, into an *AND* circuit producing the function in Boolean form of AB. This function is the input into the *NOT* circuit which gives the output, in Boolean form, of $\overline{AB}$. Note that the whole quantity of AB is complemented and not the separate variables.

3–11. Fundamental Laws and Axiomatic Expressions of Boolean Algebra

This section will give basic laws of Boolean algebra which will enable the student to simplify many Boolean expressions. These laws should be memorized so that they can easily be recalled and used as a tool for simplification.

The following is a list of these important laws. Figure 3–9 shows the switching circuit diagrams, truth tables, and logic diagram mechanizations of these laws.

I. Law of Identity
  $A = A$

II. Law of Complementarity
  1. $A\overline{A} = 0$
  2. $A + \overline{A} = 1$

III. Idempotent Law
  1. $AA = A$
  2. $A + A = A$

IV. Commutative Law
  1. $AB = BA$
  2. $A + B = B + A$

V. Associative Law
  1. $(AB)C = A(BC)$
  2. $(A+B)+C = A+(B+C)$

VI. Distributive Law
  1. $A(B+C) = (AB)+(AC)$
  2. $A+(BC) = (A+B)(A+C)$

VII. Law of Dualization (DeMorgan's Theorem)
  1. $\overline{(A+B)} = \overline{A}\,\overline{B}$
  2. $\overline{(AB)} = \overline{A}+\overline{B}$

VIII. Law of Double Negation
  $\overline{\overline{A}} = A$

IX. Law of Absorption
  1. $A(A+B) = A$
  2. $A+(AB) = A$

We shall now simplify the Boolean function $f(A, B, C, D) = AC+AD+BC+BD$ using three of the previously stated laws.

$$
\begin{aligned}
f(A,B,C,D) &= AC+AD+BC+BD \\
&= CA+CB+DA+DB \text{ by Commutative Law} \\
&= (CA+CB)+(DA+DB) \text{ by Associative Law} \\
&= [C(A+B)]+[D(A+B)] \text{ by Distributive Law} \\
f(A,B,C,D) &= (A+B)(C+D)^{*} \text{ by Distributive Law}
\end{aligned}
$$

*NOTE: In the above step, by substituting K for $A+B$ we get $K(C+D)$. Then using the Distributive Law we get $KC+KD$. Substitute $A+B$ for K and we have $C(A+B)+D(A+B)$ which gives us the step before the simplified expression $f(A, B, C, D) = (A+B)(C+B)$.

Figure 3–10 shows axiomatic expressions and their logic diagrams, truth tables, and switching circuits. Note that the 0 indicates an open circuit and the 1 indicates a closed circuit. The following are the axiomatic expressions.

1. $A+0 = A$
2. $A \cdot 0 = 0$ (The variable A may be 1 or 0.)
3. $A+1 = 1$
4. $A \cdot 1 = A$

3–12. Minterm-Maxterm Conversion

Any Boolean function may be expressed in either minterm form or maxterm form. The key to the above statement is the truth table. With the truth table of any function of n-variables, one may express this function in minterm or maxterm form.

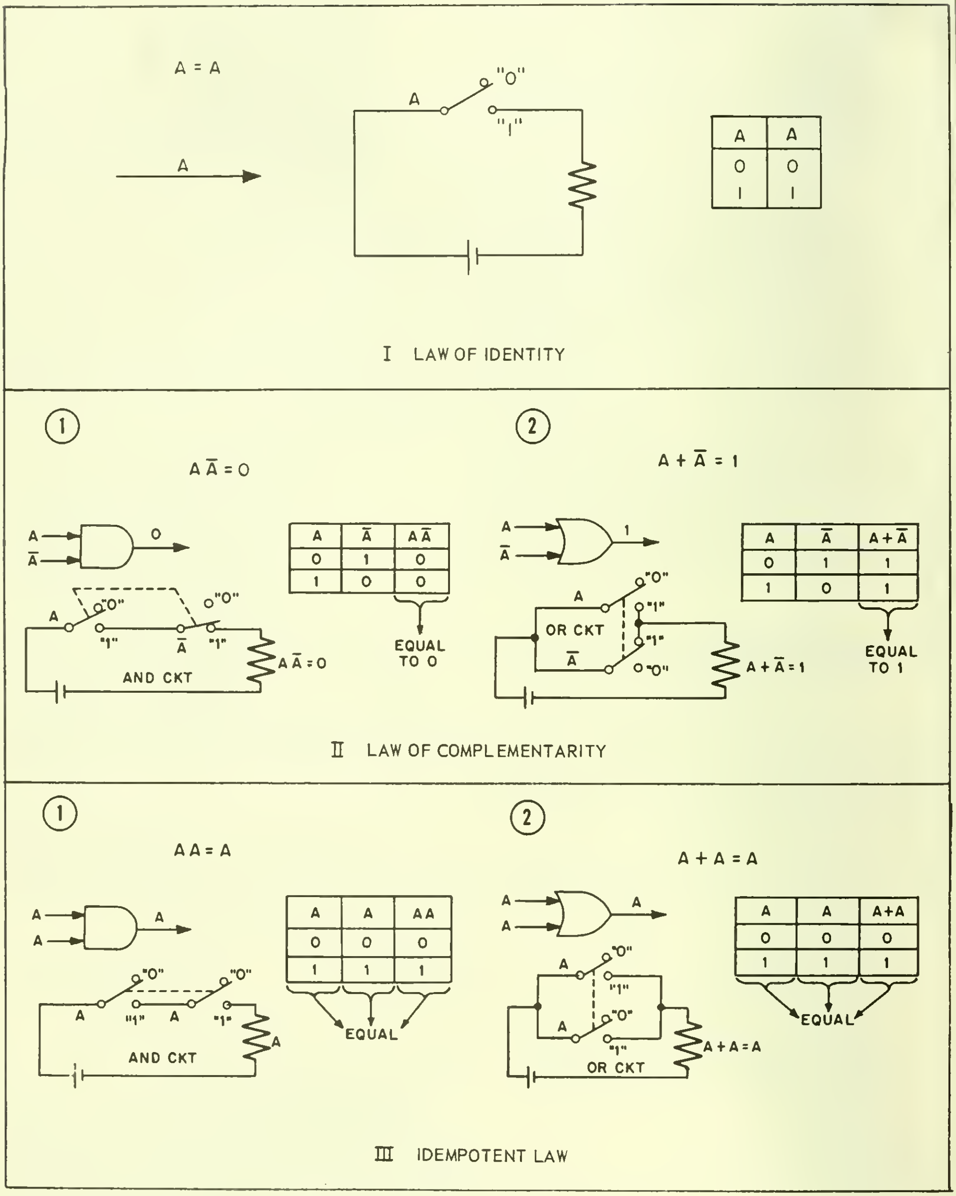
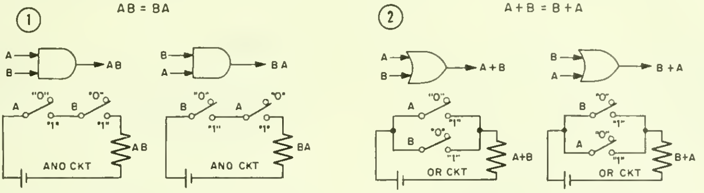


Figure 3-9.—Fundamental laws of Boolean algebra.



A	B	AB
0	0	0
0	1	0
1	0	0
1	1	1

B	A	BA
0	0	0
0	1	0
1	0	0
1	1	1

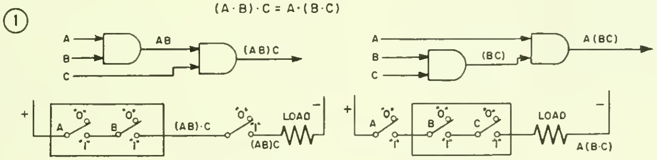
A	B	A+B
0	0	0
0	1	1
1	0	1
1	1	1

B	A	B+A
0	0	0
0	1	1
1	0	1
1	1	1

EQUAL

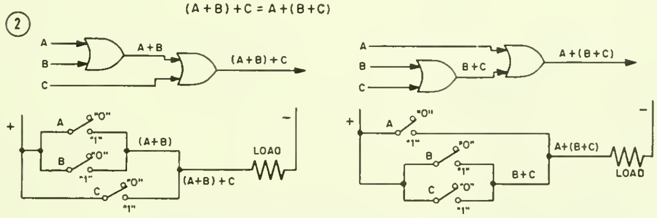
EQUAL

IV COMMUTATIVE LAW



A	B	C	AB	(AB)·C	(BC)	A(BC)
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	0	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

EQUAL



A	B	C	A+B	(A+B)+C	B+C	A+(B+C)
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	1	1	0	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

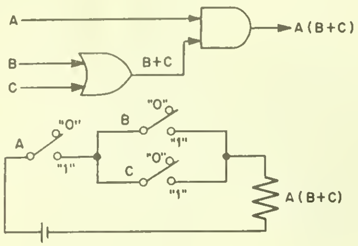
EQUAL

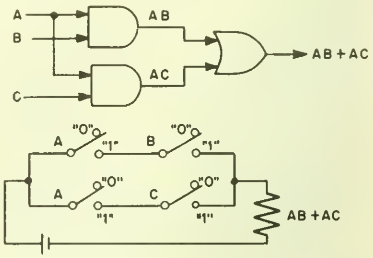
V ASSOCIATIVE LAW

Figure 3-9.—Fundamental laws of Boolean algebra—Continued.

1

$A(B+C) = (AB) + (AC)$





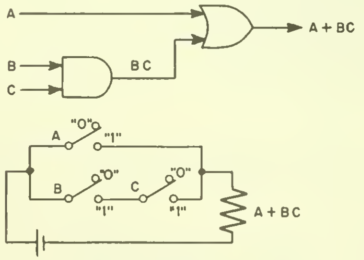
A	B	C	B+C	A(B+C)
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

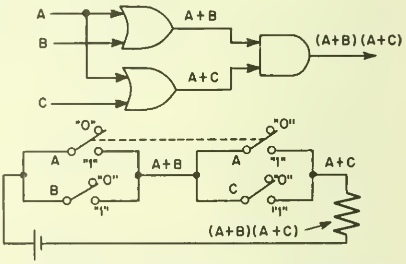
A	B	C	AB	AC	AB+AC
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	0	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	1	1

EQUAL

2

$A + BC = (A+B)(A+C)$





A	B	C	BC	A+BC
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

A	B	C	A+B	A+C	(A+B)(A+C)
0	0	0	0	0	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

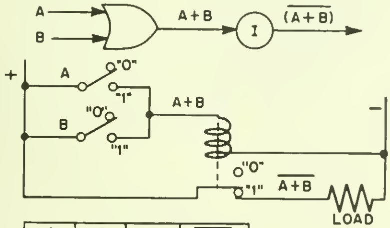
EQUAL

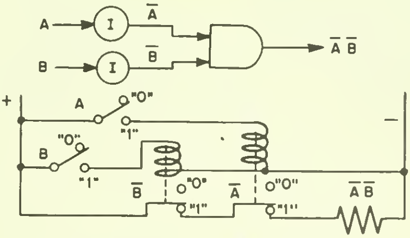
VI DISTRIBUTIVE LAW

Figure 3-9.—Fundamental laws of Boolean algebra—Continued.

1

$\overline{(A+B)} = \bar{A}\,\bar{B}$ (DE MORGAN'S THEOREM)





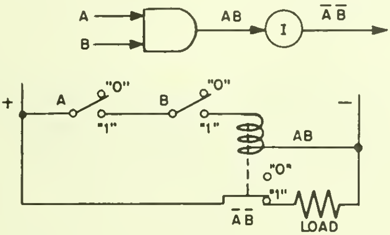
A	B	A+B	$\overline{(A+B)}$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

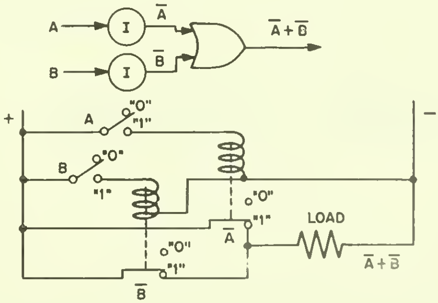
A	B	$\bar{A}$	$\bar{B}$	$\bar{A}\,\bar{B}$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

EQUAL

2

$\overline{AB} = \bar{A} + \bar{B}$ (DE MORGAN'S THEOREM)





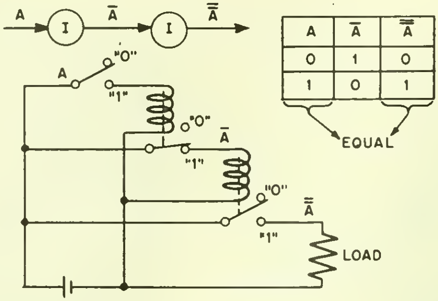
A	B	AB	$\overline{AB}$	$\bar{A}$	$\bar{B}$	$\bar{A}+\bar{B}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

EQUAL

VII LAW OF DUALIZATION

Figure 3–9.—Fundamental laws of Boolean algebra—Continued.

$\bar{\bar{A}} = A$



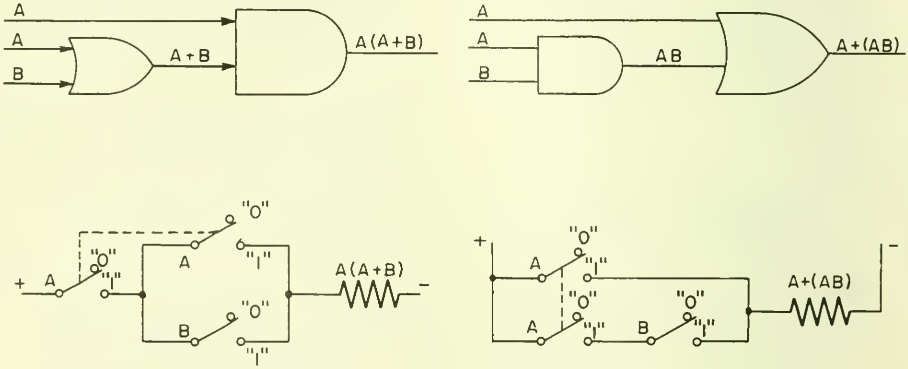
A	$\bar{A}$	$\bar{\bar{A}}$
0	1	0
1	0	1

EQUAL

VIII LAW OF DOUBLE NEGATION

(1) A (A+B) = A

(2) A + (AB) = A



A	B	A + B	A (A + B)
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

EQUAL

A	B	AB	A+(AB)
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

EQUAL

IX LAW OF ABSORPTION

Figure 3-9.—Fundamental laws of Boolean algebra—Continued.

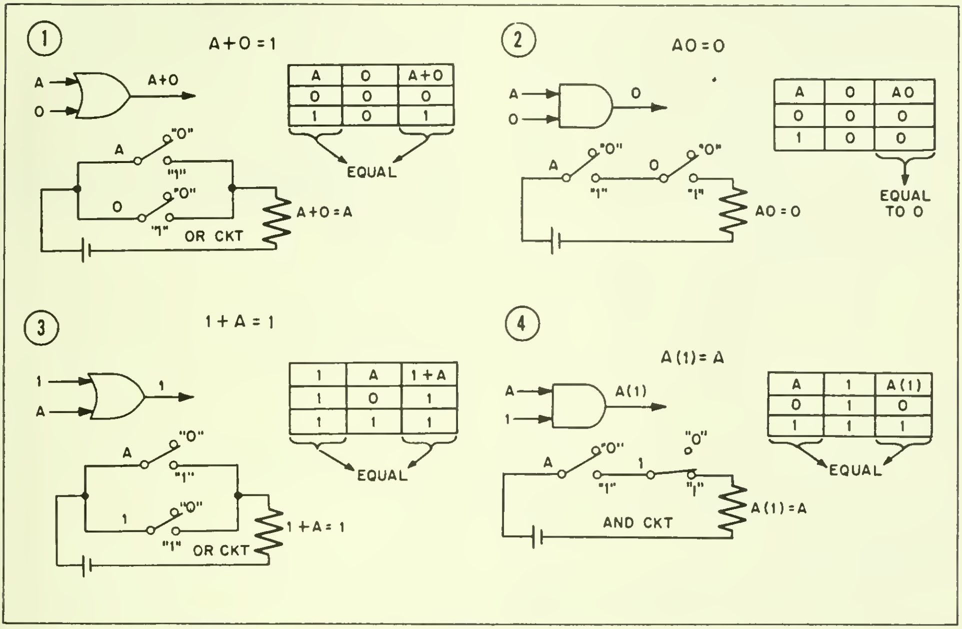


Figure 3-10.—Axiomatic expressions.

Let us consider the following truth table of a given function (table 3–1). The eight symbols f_0, f_1, f_2 . . . f_7 obtain their values from the function's value which is based on the assigned values of the variables. Each f has a corresponding minterm and maxterm. These are given in the following table (table 3–2). Note that when 0 appears in the truth table this is a complement of the variable.

TABLE 3–1.—GIVEN TRUTH TABLE.

A	B	C	$f(A, B, C)$		Complemented $f(A, B, C)$
0	0	0	0	$=f_0$	$\overline{f_0}=1$
0	0	1	1	$=f_1$	$\overline{f_1}=0$
0	1	0	0	$=f_2$	$\overline{f_2}=1$
0	1	1	0	$=f_3$	$\overline{f_3}=1$
1	0	0	0	$=f_4$	$\overline{f_4}=1$
1	0	1	1	$=f_5$	$\overline{f_5}=0$
1	1	0	0	$=f_6$	$\overline{f_6}=1$
1	1	1	1	$=f_7$	$\overline{f_7}=0$

TABLE 3–2.—CORRESPONDING MINTERMS AND MAXTERMS.

	Minterm	Maxterm
f_0	$\overline{A}\overline{B}\overline{C}$	$\overline{A}+\overline{B}+\overline{C}$
f_1	$\overline{A}\overline{B}C$	$\overline{A}+\overline{B}+C$
f_2	$\overline{A}B\overline{C}$	$\overline{A}+B+\overline{C}$
f_3	$\overline{A}BC$	$\overline{A}+B+C$
f_4	$A\overline{B}\overline{C}$	$A+\overline{B}+\overline{C}$
f_5	$A\overline{B}C$	$A+\overline{B}+C$
f_6	$AB\overline{C}$	$A+B+\overline{C}$
f_7	ABC	$A+B+C$

In order to obtain the minterm expression of $f(A, B, C)$ *OR* all f's and *AND* their corresponding minterms. Thus we arrive at the following:

$$\begin{aligned} f(A, B, C) = {} & f_0(\bar{A}\bar{B}\bar{C}) + f_1(\bar{A}\bar{B}C) \\ & + f_2(\bar{A}B\bar{C}) + f_3(\bar{A}BC) \\ & + f_4(A\bar{B}\bar{C}) + f_5(A\bar{B}C) \\ & + f_6(AB\bar{C}) + f_7(ABC) \end{aligned}$$

By substitution we get,

$$\begin{aligned} f(A, B, C) = {} & 0(\bar{A}\bar{B}\bar{C} + 1(\bar{A}\bar{B}C) \\ & + 0(\bar{A}B\bar{C} + 0(\bar{A}BC) \\ & + 0(A\bar{B}\bar{C} + 1(A\bar{B}C) \\ & + 0(AB\bar{C} + 1(ABC), \end{aligned}$$

by use of the axiomatic expressions (fig. 3–10) we obtain,

$$f(A, B, C) = \bar{A}\bar{B}C + A\bar{B}C + ABC$$

which is the minterm expression of $f(A, B, C)$. In order to check this expression, make the truth table of the expression and see if it corresponds to the given truth table.

Now, in order to obtain the maxterm expression of $f(A, B, C)$ *AND* all f's and *OR* their corresponding complemented minterms. Thus, we arrive at the following:

$$\begin{aligned} f(A, B, C) = {} & \left[f_0 + \overline{(\bar{A}\bar{B}\bar{C})}\right]\left[f_1 + \overline{(\bar{A}\bar{B}C)}\right] \\ & \times \left[f_2 + \overline{(\bar{A}B\bar{C})}\right]\left[f_3 + \overline{(\bar{A}BC)}\right] \\ & \times \left[f_4 + \overline{(A\bar{B}\bar{C})}\right]\left[f_5 + \overline{(A\bar{B}C)}\right] \\ & \times \left[f_6 + \overline{(AB\bar{C})}\right]\left[f_7 + \overline{(ABC)}\right] \end{aligned}$$

By substitution and use of the Law of Dualization (DeMorgan's Theorem) we get,

$$\begin{aligned} f(A, B, C) = {} & [0 + (A + B + C)] \\ & \times [1 + (A + B + \bar{C})] \\ & \times [0 + (A + \bar{B} + C)] \\ & \times [0 + (A + \bar{B} + \bar{C})] \\ & \times [0 + (\bar{A} + B + C)] \\ & \times [1 + (\bar{A} + B + \bar{C})] \\ & \times [0 + (\bar{A} + \bar{B} + C)] \\ & \times [1 + (\bar{A} + \bar{B} + \bar{C})] \end{aligned}$$

By use of the axiomatic expressions we arrive at:

$$\begin{aligned} f(A, B, C) = {} & (A + B + C)(A + \bar{B} + C) \\ & \times (A + \bar{B} + \bar{C})(\bar{A} + B + C) \\ & \times (\bar{A} + \bar{B} + C) \end{aligned}$$

which is the maxterm expression of $f(A, B, C)$. In order to check this expression, make the truth table of the expression and see if it corresponds to the given truth table.

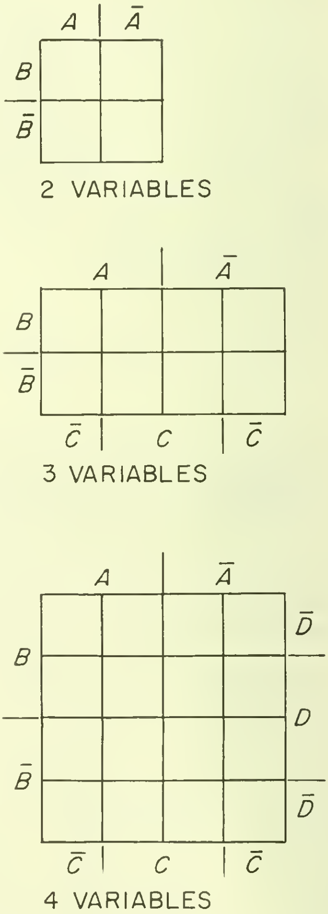


Figure 3–11.—The Veitch diagram (2, 3, 4, variables).

3–13. The Veitch Diagram

In previous sections we have discussed various ways to represent Boolean functions. These were, as a minterm, maxterm, truth table, Venn diagram, switching circuit, and a logic diagram mechanization. Another form of function representation is the Veitch diagram. This section will discuss the construction of the Veitch diagram.

Veitch diagrams provide a very quick and easy way for finding the simplest logical equation needed to express a given function. This will be discussed in detail in the next chapter.

Veitch diagrams for two, three, four, five or more variables are readily constructed. Any number of variables may be plotted on a Veitch diagram, though the diagrams are difficult to use when more

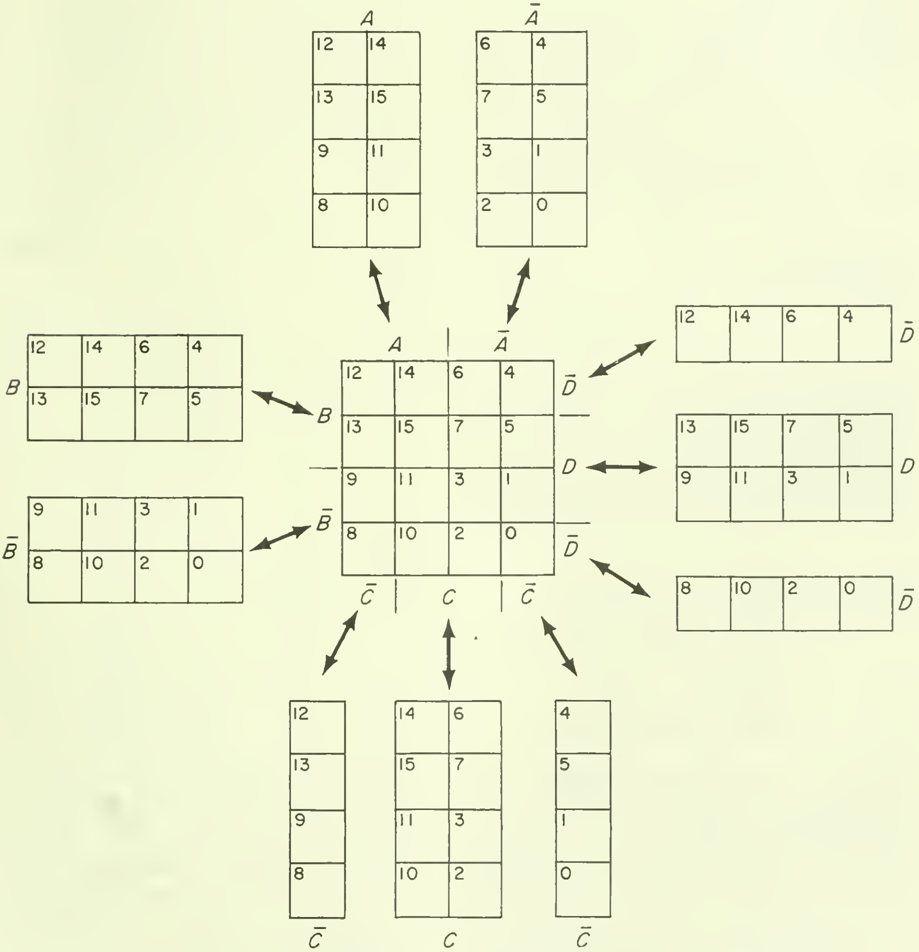


Figure 3–12.—Exploded Veitch diagram.

than four variables are involved. The Veitch diagrams for two through four variables are illustrated in figure 3–11.

Since each variable has two possible states (0 and 1), the number of squares needed is the number of possible states (two) raised to a power dictated by the number of variables. Thus, for four variables the Veitch diagram must contain 2^4 or 16 squares. Five variables require 2^5 or 32 squares. An eight-variable Veitch diagram needs 2^8 or 256 squares—a rather unwieldy diagram. If it becomes necessary to simplify logical equations containing more than six variables, other methods of simplification are available and will be discussed in the next chapter.

An exploded view of a four-variable Veitch diagram is shown in figure 3–12. Notice the division marks which divide the diagram into labeled columns and rows. The location of each square represents the combination of the variables labeling each row and column.

To illustrate the plotting of the Veitch diagram, the Boolean equation

$$f(A, B, C, D) = ABC + AB\overline{D} + A\overline{C} + \overline{A}\overline{B}\overline{C}\overline{D} + \overline{A}C$$

will be used. Since there are four variables, a four-variable Veitch diagram is needed. There will be $2^4 = 16$ squares. (Figure 3–12 is a four-variable Veitch diagram.) The Boolean function is now plotted term-by-term on the Veitch diagram. This is accomplished by placing a "1" in each square representative of the term, as follows: (fig. 3–13).

$$f(A, B, C, D) = ABC \text{......squares 14 and 15}$$
$$= +AB\overline{D} \text{......squares 12 and 14}$$
$$= +A\overline{C} \text{......squares 12, 13, 9, and 8}$$
$$= +\overline{A}\overline{B}\overline{C}\overline{D} \text{......square 0}$$
$$= +\overline{A}C \text{......squares 6, 7, 3, and 2}$$

The Veitch diagram now looks like figure 3–13. Notice that when representing a term which does not contain all the variables, (A, B, C, and D) the plotting will represent both the negated and non-negated forms of the variables which were missing from the term. (Refer to the plotting of AC in the above explanation.)

The decimal numbers in the upper left hand corner of each square in figure 3–12 and figure 3–13 are the numbers corresponding to the following truth table.

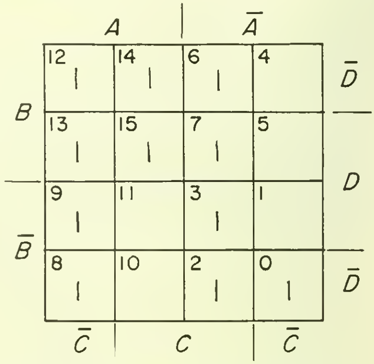


Figure 3–13.—Veitch diagram of a Boolean function.

TABLE 3–3.—TRUTH TABLE OF $f = ABC + AB\overline{D} + A\overline{C} + \overline{A}\overline{B}\overline{C}\overline{D} + \overline{A}C$

Decimal Equivalents of the Binary Represented Variables	A	B	C	D	f
0	0	0	0	0	1
1	0	0	0	1	0
2	0	0	1	0	1
3	0	0	1	1	1
4	0	1	0	0	0
5	0	1	0	1	0
6	0	1	1	0	1
7	0	1	1	1	1
8	1	0	0	0	1
9	1	0	0	1	1
10	1	0	1	0	0
11	1	0	1	1	0
12	1	1	0	0	1
13	1	1	0	1	1
14	1	1	1	0	1
15	1	1	1	1	1

The above truth table contains all possible combinations of the variables. These combinations are arranged in a binary counting sequence as shown by their decimal equivalents. The decimal equivalents are placed in the left hand corners of the squares in a Veitch diagram which represent the combinations of the variables. Notice that only the combinations where "f" equals "1" are plotted on a Veitch diagram (fig. 3–13).

CHAPTER 4
SIMPLIFICATION TECHNIQUES AND APPLICATIONS

4–1. Introduction

A logical designer may arrive at his Boolean function by means of the minterm or maxterm expression derived from a truth table. This form may often be simplified. By simplified, we mean that another expression may be arrived at that will represent the same function but may be constructed with less equipment. For example, the designer may arrive at the function $f(A, B, C) = A\overline{B} + B\overline{C} + \overline{B}C + \overline{A}B$. This function can be simplified to give

$$f(A, B, C) = A\overline{B} + B\overline{C} + \overline{A}C$$

which may be easier to construct.

When describing Boolean functions it is often necessary to identify them as to their ORDER. The order is defined, for example, so that the cost of the logic circuit may be determined without constructing the circuit. Higher order expressions cost more to construct.

To determine the order of a Boolean expression, one must first inspect the quantity within the parentheses. If this quantity contains only an *AND* operation(s), or only an *OR* operation(s), this quantity is first-order. If the quantity contains both an *AND* and an *OR* operation(s), it is considered a second-order quantity.

The next step is to consider the relationship of the quantity within the parentheses and the other variables within the brackets of the expression. Again, if the parenthesized quantity is combined with the other bracketed variables with either an *AND* operation(s) or an *OR* operation(s), the order is increased accordingly. This process is continued until the final order of the expression is obtained.

The following examples will show the student the order-determining process.

Example 4–1.

Find the order of the Boolean expression
$f(A, B, C, D, E, F, G) = [(AB + C)D + E]F + G.$

(1) Consider the parenthesized quantity $(AB + C)$.
  a. Does it contain *AND* operation(s)?
    Ans. Yes!
  b. Does it contain *OR* operation(s)?
    Ans. Yes!
  c. Conclusion: 2nd-order quantity.

(2) Consider the bracketed quantity

$$[(AB + C)D + E].$$

  a. Is the parenthesized quantity combined with the other variables with *AND* operation(s)?
    Ans. Yes!
  b. Combined with *OR* operation(s)?
    Ans. Yes!
  c. Conclusion: Increase order of parenthesized quantity by 2 which yields a 4th-order bracketed quantity.

(3) Consider the braced quantity

$$\{[(AB + C)D + E]F + G\}.$$

  a. Is the bracketed quantity combined with the other variables with *AND* operation(s)?
    Ans. Yes!
  b. Combined with *OR* operation(s)?
    Ans. Yes!
  c. Conclusion: Increase order of bracketed quantity by 2 which yields a 6th-order, braced, final quantity.

Thus $f = [(AB + C)D + E]F + G$ is a 6th-order Boolean expression. Note that the number of variables has no relationship to the order of a Boolean expression.

Example 4–2

Find the order of the Boolean function

$$f(A, B, C, D, E, F, G) = [(AB)(CD + EF) + G].$$

We will proceed to the solution with limited discussion.

(1) Consider $(CD+EF)$. (Note: Start process with the highest order parenthesized quantity.)
   a. *AND* operation(s)?
      Ans. Yes!
   b. *OR* operation(s)?
      Ans. Yes!
   c. Conclusion: 2nd-order.

(2) Consider $[(AB)(CD+EF)+G]$.
   a. *AND* operation(s)?
      Ans. Yes!
      ((AB) $(CD+EF)$ is an *AND* operation.)
   b. *OR* operation(s)?
      Ans. Yes!
   c. Conclusion: Increase 2nd-order parenthesized quantity by 2 which yields a 4th-order, bracketed, final quantity.

Thus $f=[(AB)(CD+EF)+G]$ is a fourth-order Boolean expression.

In this chapter, we will concentrate for the most part on first- and second-order expressions. As yet, no one has been able to devise a simplification procedure for any order Boolean expression. Certain procedures work best with certain order expressions, and first- and second-order expressions are the most common.

4–2. Application of Previously Learned Basic Laws and Axiomatic Expressions

The first process of simplification to be discussed requires the student to employ ingenuity, judgment, and experience to simplify an expression by applying appropriate laws and axiomatic expressions of Boolean algebra. For example, the left-hand side of the following expressions requires more elements of construction than the right-hand side. Therefore, if the left-hand side appears in some function, a simplification may be arrived at if it is replaced by the right-hand side.

1. $A\bar{A}=0$
2. $AA=A$
3. $A+\bar{A}=1$
4. $A+A=A$
5. $A+1=1$
6. $A1=A$
7. $A+0=A$
8. $A0=0$
9. $A+AB=A$
10. $A+\bar{A}B=A+B$
11. $A(A+B)=A$
12. $A(\bar{A}+B)=AB$
13. $(A+B)(A+C)=A+BC$
14. $AB+AC=A(B+C)$
15. $AC+\bar{A}B+BC=(A+B)(\bar{A}+C)=AC+\bar{A}B$

The application of these expressions may not be obvious. It is usually necessary to rearrange, and even modify, the original function before any of the above expressions can be used. For example, it might be helpful to "*OR*" $X\bar{X}$ (axiomatic term equal to 0) to the original function, or to "*AND*" it by $(X+\bar{X})$, (axiomatic term equal to 1) if an X can be chosen which eventually can be eliminated.

Also, DeMorgan's Theorem may be a useful tool in simplifying. DeMorgan's Theorem is stated as follows:

To negate a Boolean expression:

1. Interchange *OR*s for *AND*s and *AND*s for *OR*s observing all parentheses or brackets present and implied.
2. Negate each variable of the new expression.

For example: given $f=(A+B)D$ then

$$\bar{f}=\overline{(A+B)D}=\overline{AD+BD}=(\bar{A}+\bar{D})(\bar{B}+\bar{D})=\overline{AB}+\bar{D}$$

or given $f=A+B$ then $\bar{f}=\overline{A+B}=\overline{AB}$

or given $f=AB$ then $\bar{f}=\overline{AB}=\bar{A}+\bar{B}$

Following are some examples to illustrate the use of previously learned basic laws and axiomatic expressions in simplifying Boolean functions.

Example 4–3.
Simplify $f=A\bar{B}+D+\bar{A}\bar{D}C+B\bar{D}C$

Applying #14 to the last two terms, we have:

$$f=A\bar{B}+D+(\bar{D}C)(\bar{A}+B)$$

Applying #10 to the last two terms by substitution of $D+C$ for $D+\bar{D}C$ we find:

$$f=A\bar{B}+D+C(\bar{A}+B)$$

Using DeMorgan's theorem on the last term we find:

$$f=A\bar{B}+D+C(\overline{\overline{\bar{A}+B}})$$

$$f=A\bar{B}+D+C(\overline{A\bar{B}})$$

Applying #10 again by substituting $A\bar{B}+C$ for $A\bar{B}+(C)(\overline{A\bar{B}})$ we finally arrive at:

$$f=A\bar{B}+D+C$$

Example 4–4.
Simplify $f=AB+C+\bar{A}B\bar{C}+\bar{A}\bar{B}\bar{C}$

Applying #14 to the last two terms we have:

$$f=AB+C+\bar{C}(\bar{A}B+\bar{A}\bar{B}).$$

Then applying #10 to the last two terms by substituting C of the example for A of #10 and $(\bar{A}B+\bar{A}\bar{B})$ of the example for B of #10 we find:

$$f=AB+C+\bar{A}B+\bar{A}\bar{B}$$

Using #4 by substituting $\bar{A}B$ of the example for A in #4 and noting any number of terms may be added if the term is already present in the original equation; we can substitute $\bar{A}B+\bar{A}B$ for $\bar{A}B$ in the example as follows:

$$f=AB+\bar{A}B+\bar{A}B+\bar{A}\bar{B}+C$$

Again we use #14 by factoring B from the first two terms and $\bar{A}$ from the third and fourth term to give:

$$f=B(A+\bar{A})+\bar{A}(B+\bar{B})+C$$

Finally, using #3, substituting 1 for both $(A+A)$ and $(B+\bar{B})$, we find the simplified form:

$$f=B+\bar{A}+C$$

Example 4–5.
Simplify $f=A\bar{B}+B\bar{C}+\bar{B}C+\bar{A}B$

There is no obvious simplification of this expression. But, if the last two terms are logically multiplied (*AND* operation) by $(A+\bar{A})$ and $(C+\bar{C})$, respectively, the resulting terms may be simplified as follows: According to rule #3 logically multiply the last two terms by the equivalent of 1,

$$f=A\bar{B}+B\bar{C}+\bar{B}C(A+\bar{A})+\bar{A}B(C+\bar{C})$$

Logically multiply the last two terms by their respective common factors according to #14.

$$f=A\bar{B}+B\bar{C}+A\bar{B}C+\bar{A}\bar{B}C+\bar{A}BC+\bar{A}B\bar{C}$$

Rearranging the order of terms,

$$f=A\bar{B}+A\bar{B}C+B\bar{C}+\bar{A}B\bar{C}+\bar{A}\bar{B}C+\bar{A}BC.$$

Using #14 again, combine term pairs having common factors,

$$f=A\bar{B}(1+C)+B\bar{C}(1+\bar{A})+\bar{A}C(\bar{B}+B).$$

Finally, by applying #15, to the first two terms and #3 to the last term we find the simplified form:

$$f=A\bar{B}+B\bar{C}+\bar{A}C$$

Note that in the previous examples, the final expression is simpler than the original. For a beginning student in Boolean algebra, the above use of basic laws and axiomatic expressions to simplify a Boolean function may seem difficult and tedious. But to an expert in Boolean algebra the simplifications are fairly obvious.

4–3. Veitch Diagrams

We shall now review the plotting procedures to construct a Veitch diagram. The number of variables in a given Boolean function determines the number of squares in a Veitch diagram. The number of squares needed is the number of possible states (two) raised to a power dictated by the number of variables. Thus, for two variables the Veitch diagram must contain 2^2 or 4 squares. Three variables require 2^3 or 8 squares (chap. 3, fig. 3–11).

To plot the logical function on the Veitch diagram, place a "1" in each square representative of the term. (Chap. 3, fig. 3–13.)

Figure 4–1 shows the Veitch diagram of $f=ABC+AB\bar{D}+A\bar{C}+\bar{A}\bar{B}\bar{C}\bar{D}+\bar{A}C$. We will use this expression in discussing the use of the Veitch diagram in simplification.

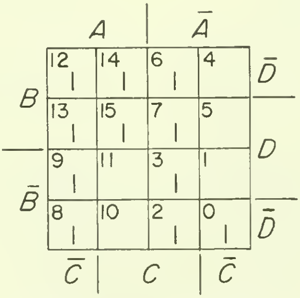


Figure 4–1.—The Veitch diagram of $f=ABC+AB\bar{D}+A\bar{C}+\bar{A}\bar{B}\bar{C}\bar{D}+\bar{A}C$.

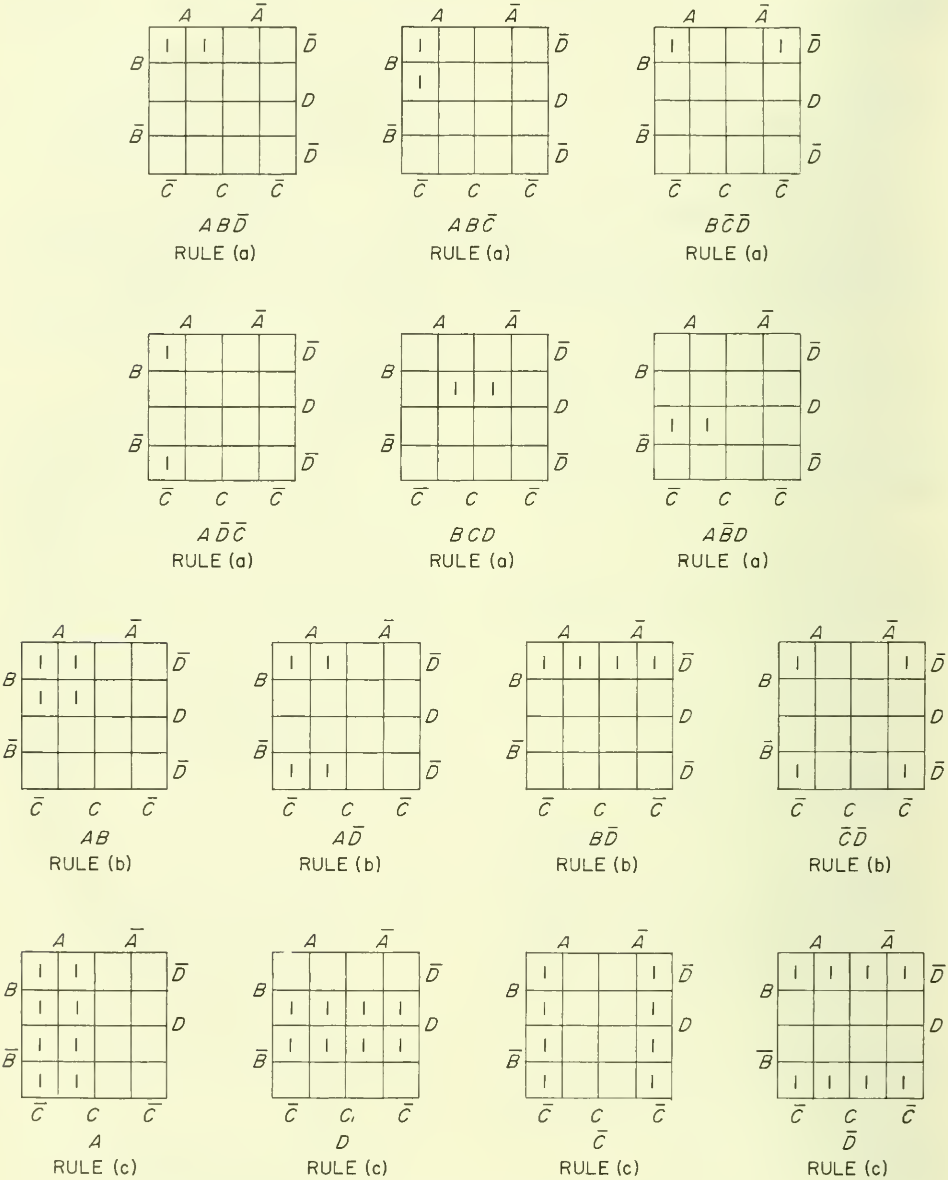


Figure 4-2.—Veitch combinations.

To obtain the simplified logical equation from a Veitch diagram of four variables, observe the following rules: (Fig. 4–2.)

a. If 1's are located in adjacent squares or at opposite ends of any row or column, one of the variables may be dropped. Note that those variables that remain must not yield a Boolean product of zero.
b. If any row or column of squares, any block of four squares, or the four end squares of any adjacent rows or columns, or the four corner squares are filled with 1's two of the variables may be dropped.
c. If any two adjacent rows or columns, the top and bottom rows, or the right and left columns are completely filled with 1's, three variables may be dropped.
d. To reduce the original equation to its simplest form, sufficient simplification must be made until all 1's have been included in the final equation. 1's may be used more than once, and the largest possible combinations of 1's in groups of 8, 4, or 2 should be used.

To proceed with the simplification of $f=ABC+AB\overline{D}+A\overline{C}+\overline{A}\overline{B}\overline{C}\overline{D}+\overline{A}C$:

1. Squares 12, 13, 9, and 8 may be combined using rule (b) to yield $A\overline{C}$.
2. Squares 6, 7, 3, and 2 may be combined using rule (b) to yield $\overline{A}C$.
3. Squares 12, 14, 13, and 15 may be combined using rule (b) to yield AB.
4. Squares 0 and 2 may be combined using rule (a) to yield $\overline{A}\overline{B}\overline{D}$.

To keep track of the squares combined, draw loops around the combined squares. Doing this, the Veitch diagram is modified as in figures 4–3.

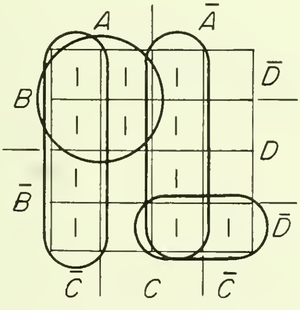


Figure 4–3.—Combined squares of Veitch diagram.

All 1's have been used at least once therefore, the Boolean expression can now be written in its simplest form as

$$f=AB+A\overline{C}+\overline{A}C+\overline{A}\overline{B}\overline{D}.$$

A Veitch diagram provides a convenient way for finding the complement of a logical expression. This is done by plotting the original equation on a Veitch diagram, and then plotting on another Veitch diagram an expression which has "ones" everywhere the original expression does not have "ones." An example will illustrate the procedure.

Example 4–6.

If $f=ABC$ what is $\overline{f}$? $\overline{f}=\overline{ABC}=\overline{A}+\overline{B}+\overline{C}$ (DeMorgan's theorem). A three-variable Veitch diagram should give the same answer. The original expression is first plotted as figure 4–4.

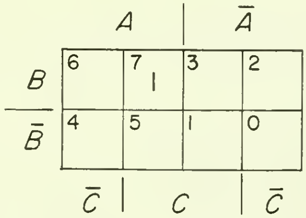


Figure 4–4.—Veitch diagram of $f=ABC$.

On another Veitch diagram, all squares which do not have a "one" in figure 4–4, are assigned a "one" as figure 4–5.

Now the expression for $\overline{f}$ can be written from figure 4–5. Squares 3, 2, 1, and 0 combine to form $\overline{A}$; squares 4, 5, 1, and 0 combine to form $\overline{B}$; and

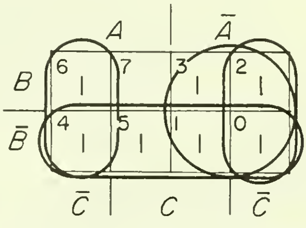


Figure 4–5.—Veitch diagram of $\overline{f}=\overline{ABC}$.

squares 6, 4, 2, and 0 combine to form $\overline{C}$. Therefore, the expression for $\overline{f}$ is,

$$\overline{f}=\overline{A}+\overline{B}+\overline{C}$$

which agrees with the result obtained by directly complementing the original expression.

4–4. Application of Chapters 2, 3, and 4

We shall now use information from chapters 2 and 3, and this chapter to present an application of Boolean algebra and number systems to certain aspects of a digital computer design (specifically an adder and a subtractor logic diagram mechanization).

The first step is to identify the problem and select a known solution. Take a sample problem and find the solution. Look back and see what processes (addition, multiplication, subtraction, division) were used to obtain the solution. Compare these human processes with those of a machine.

In our world of numbers, we have both positive and negative numbers. It would be beneficial if our machine could represent plus (+) and minus (−) quantities. To do this we will use the SIGN BIT NOTATION.

Sign bit notation employing the binary number system uses 1 to represent a minus number and 0 to represent a plus number. If a number is negative each digit must be ones complemented (ch. 2, sec. 2–5, *C*) and the sign bit is added in the proper column(s). The following example will help the student to understand sign bit notation.

Example 4–7.

Represent -4_{10}, and -2_{10} in binary sign bit notation.

A. $-4_{10}=-100_2$

$-100_2=1\,0\,1\,1$

This one is a result of one's complementing.

This one is a result of one's complementing.

This zero is a result of one's complementing.

This one equals minus using sign bit notation in sign bit column.

$\therefore -4_{10}=1\,0\,1\,1_2$ using sign bit notation which employs one's complementing.

B. $-2_{10}=-0\,0\,0\,1\,0_2$ (employing a six bit register with the minus sign included as one bit)

$-0\,0\,0\,1\,0_2=1\,1\,1\,1\,0\,1_2$

one's complement

sign bit

$\therefore -2_{10}=1\,1\,1\,1\,0\,1_2$ using sign bit notation which requires one's complementing when employing a six bit register.

Such a system of notation demands end-around-carry (chap. 2, Sec. 2–5,C) or end-around-borrow (to be explained) when performing either additive or subtractive processes.

A digital computer has to use either an additive or a subtractive process. Most computers use a subtractive process because it has an inability to get a negative zero (described later). Following are examples to show the student both the additive and subtractive processes used in digital computers.

Example 4–8.

A. Additive $A+B$ (using sign bit, end-around-carry)

$$A=-21_{10},\ B=+5_{10}$$

The number of bits (digit columns in the number) required is equal to one for the sign bit plus the number of bits required to represent 21 in the binary equivalent (because 21 is the larger absolute number).

$\therefore$ $\quad -21_{10}=-1\,0\,1\,0\,1_2=1\,0\,1\,0\,1\,0$

and $\quad +5_{10}=+0\,0\,1\,0\,1_2=0\,0\,0\,1\,0\,1$

then

$$\begin{array}{rl}
A=101010 & \\
+ & \\
B=000101 & \\
\hline
HA=101111 & \text{(half add)}\\
CR=00000- & \text{(carry)}\\
\hline
SUM=101111 & \text{(This equals sum for there were no carries to be considered for the } HA \text{ and } CR \text{ operation)}
\end{array}$$

$\therefore \quad 101111=-10000_2=-16_{10}$

B. Additive $A+B$ (using sign bit, end-around-carry)

$$A=-8,\quad B=-15$$
$$-15_{10}=-1111_2=10111$$

(This determines the number of bits because it has the larger absolute value.)

$$-8_{10}=-10000_2=10111$$

$$\begin{array}{rl}
\therefore\ A= & 10111 \\
+ & \\
B= & 10000 \\ \hline
HA= & 00111 \\
CR= & 10000- \\ \hline
HA= & 1\,00111 \\
EAC= & \quad\ \ 1 \quad \text{(end-around-carry)} \\ \hline
HA= & 00110 \\
CR= & 0001- \\ \hline
HA= & 00100 \\
CR= & 001-- \\ \hline
HA= & 00000 \\
CR= & 01--- \\ \hline
HA= & 01000 \\
CR= & 0---- \\ \hline
SUM= & 01000=+8
\end{array}$$

But this is incorrect! The problem used two numbers, which when added produced a sum that exceeded the modulus of the register (needed a larger bit register). How could a computer know? By checking the two sign bits of the original numbers to be added. If they are both zeros then the final result should be a positive number. If they are both one's then the final result should be negative. In the above case the final sum is known to be in error because

(1) two negative numbers were added,

(2) the result was a positive number.

The sum would then be one's complemented by the computer and the absolute value without the sign could be read with the negative sign indicated at some designated place on the read-out device.

∴ in our case,

(one's complement)
$$01000=10111=23_{10}$$

and the final answer is

$$-23_{10}.$$

Example 4–9

A. Subtractive A-B (using sign bit notation, end-around-borrow).

In example 4–8 we used the additive process of "half-add" and "carry" operations. But in the process of subtraction, the operations of "half-subtract" and "borrow" are utilized.

The computer technique of subtraction includes the operations of half-subtract, borrow, and final difference. The first logical conclusion in subtraction is the examination of the minuend and the subtrahend to determine the half-subtract operation. The half-subtract operation consists first of the subtraction of the numbers: neglecting to record the digit change in the minuend due to borrowing. The second logical conclusion is determining the number of borrows and noting their place in reference to the half-subtract conclusion. The third logical conclusion is finding the final difference as a result of the half-subtract and the borrow conclusions.

$$A=-21_{10}\quad B=+5_{10}$$
$$-21_{10}=-10101_2=101010$$

(using sign bit notation which requires one's complementing; the larger absolute value ($A=-21_{10}$) determines the number of bits).

$$+5=+101_2=000101$$

$$\begin{array}{rl}
\therefore\ A= & 101010 \\
- & \\
B= & 000101 \\ \hline
HS= & 101111 \quad \text{(half-subtract)} \\
BR= & 00101- \quad \text{(borrows)} \\ \hline
HS= & 100101 \quad \text{(half-subtract)} \\
BR= & 0000-- \quad \text{(borrows)} \\ \hline
DIF= & 100101 \quad \text{(difference, for there were no borrows to be considered for the preceding HS and BR operation)}
\end{array}$$

$$\therefore\ 100101=-11010_2=-26_{10}$$

B. Subtractive $A-B$

(using sign bit notation, end-around-borrow).

In the additive operation, when there arose an extra column digit in a carry operation past the sign bit column, we used the process of "end-around-carry." In the subtractive operation, if an extra digit appears in the column to the left of the sign bit column, we "end-around-borrow" (*EAB*) it to the units column in its own separate process. The following problem will illustrate this process of "end-around-borrow."

$$A = -5 \qquad B = -2$$

$$A = -5_{10} = -101_2 = 1010$$

(using sign bit notation)

$$B = -2_{10} = -10 = 1101$$

$$\begin{array}{rr}
\therefore\quad A = & 1010 \\
 & - \\
B = & 1101 \\ \hline
HS = & 0111 \\
BR = & 101- \\ \hline
HS = & 1101 \\
BR = & 100\!-\!\!- \\ \hline
HS = & 11101 \\
EAB = & 1 \\ \hline
HS = & 1100 \\
BR = & 0000 \\ \hline
DIF = & 1100
\end{array} \quad \text{(End-around-borrow)}$$

$$\therefore \quad 1100_2 = -011 = -3_{10}$$

Now to exemplify the advantage of the subtractive process over the additive process, we will work a sample problem and obtain the negative zero.

Example 4–10.

By the additive process, add -7 and $+7$ using end-around-carry, and sign bit notion.

$$-7_{10} = -111_2 = 1000$$

$$+7_{10} = +111_2 = 0111$$

(using sign bit notation)

$$\begin{array}{r}
A = 0111 \\
+ \\
B = 1000 \\ \hline
HA = 1111 \\
CR = 0000 \\ \hline
SUM = 1111
\end{array}$$

$$\therefore \; 1111_2 = -000_2 = -000_{10}$$

Notice the negative zero. Due to network setup in a digital computer, it is time consuming to work with a negative zero. Thus, the subtractive process is used more widely in computer design for it will never arrive at a negative zero. (See example 4–11.)

Example 4–11.

By the subtractive process, subtract $+7$ from $+7$ using end-around-borrow, and sign bit notation.

$$+7 = +111_2 = 0111 \text{ using sign bit notation}$$

$$\begin{array}{r}
A = 0\,1\,1\,1 \\
- \\
B = 0\,1\,1\,1 \\ \hline
HS = 0\,0\,0\,0 \\
BR = 0\,0\,0\,0 \\ \hline
DIF = 0\,0\,0\,0
\end{array}$$

$$\therefore 0000_2 = +000_2 = +000_{10}$$

After selection of one of the above processes (additive or subtractive) an example should be processed and appropriate designations for level, register (*A* or *B*), intermediate and final solution tagged. Next, each intermediate and final digit position process expression (Boolean) must be written, and simplified. Once the simplified Boolean expressions are known, the registers and logical circuits can be tied together to produce the desired computer network.

We shall now consider both the additive process and the subtractive process using a three-level register and sign bit processing.

Operation: $A + B$ to C (The plus (+) denotes the additive process.)

Truth table for half-add/carry process. (The conditions that A and B can assume are indicated in the first two rows.)

A	0	0	1	1
B	0	1	0	1
Half-add	0	1	1	0
Carry	0	0	0	1

Sample problem:

$$
\begin{array}{l}
A = 1_3\,0_2\,1_1\,1_0 = -100 = -4_{10} \\
+ \\
B = 1_3\,1_2\,0_1\,1_0 = -010 = -2_{10} \\
\hline
HA = 0_3\,1_2\,1_1\,0_0 \\
CR = 0_2\,0_1\,1_0\,1_{eac} \text{ where } CR_3 = EAC \\
\hline
HA = 0_3\,1_2\,0_1\,1_0 \\
CR = 0_2\,1_1\,0_0\,0_{eac} \\
\hline
HA = 0_3\,0_2\,0_1\,1_0 \\
CR = 1_2\,0_1\,0_0\,0_{eac} \\
\hline
HA = 1_3\,0_2\,0_1\,1_0 \\
CR = 0_2\,0_1\,0_0\,0_{eac} \\
\hline
SUM = 1_3\,0_2\,0_1\,1_0 = -110 = -6_{10}
\end{array}
$$

Each subscript denotes level and each half-add (*HA*) process denotes intermediate sums. The carry (*CR*) always has an end-around-carry (*eac*) level indicated in the extreme right-hand column and the subscripts denote at which level the carry originated.

Boolean expressions:

Individual

A. Half-add operation

(1) $HA_0 = A_0\overline{B}_0 + \overline{A}_0B_0$

This expression is derived from the preceding truth table and is read from left to right as follows: The half-add operation at the zero level (HA_0) is equal to a one (1) if it should happen that A_0 is equal to a one and B_0 is not a one or A_0 is not a one and B_0 is a one.

(2) $HA_1 = A_1\overline{B}_1 + \overline{A}_1B_1$

(3) $HA_2 = A_2\overline{B}_2 + \overline{A}_2B_2$

(4) $HA_3 = A_3\overline{B}_3 + \overline{A}_3B_3$

B. Carry operation

(1) $CR_0 = A_0B_0 + (EAC)(A_0 + B_0)$.

The first term of this expression is based on the preceding truth table and is read from left to right as follows: The carry operation at the zero level (CR_0) is equal to a one (1) if it should happen that A_0 is equal to a one and B_0 is equal to a one.

The second term of this expression is based on the preceding truth table and the following examples.

$$
\begin{array}{lcl}
A = 1_3 0_2 1_1 1_0 & \quad & A = 1_3 0_2 1_1 0_0 \\
+ & & \\
B = 1_3 1_2 0_1 0_0 & & B = 1_3 1_2 0_1 1_0 \\
\hline
HA = 0_3 1_2 1_1 1_0 & & HA = 0_3 1_2 1_1 1_0 \\
CR = 0_2 0_1 0_0 1_{eac} & & CR = 0_2 0_1 0_0 1_{eac} \\
\hline
HA = 0_3 1_2 1_1 0_0 & & HA = 0_3 1_2 1_1 0_0 \\
CR = 0_2 0_1 1_0 0_{eac} & & CR = 0_2 0_1 1_0 0_{eac} \\
\hline
\textit{and so forth} & & \textit{and so forth}
\end{array}
$$

The second term of the expression is read from left to right as follows: The carry operation at the zero level (CR_0) is equal to a one (1) if it should happen that the end-around-carry (*EAC*) is equal to a one and A_0 is one or B_0 is one. Notice that the end-around-carry (*eac*) can also have the designation CR_3 in the preceding example.

(2) $CR_1 = A_1B_1 + (CR_0)(A_1 + B_1)$

(3) $CR_2 = A_2B_2 + (CR_1)(A_2 + B_2)$

Substituting (1) and (2) in (3)

$$
\begin{aligned}
CR_2 = A_2B_2 &+ (A_1B_1)(A_2 + B_2) \\
&+ (A_0B_0)(A_1 + B_1)(A_2 + B_2) \\
&+ (EAC)(A_0 + B_0)(A_1 + B_1)(A_2 + B_2)
\end{aligned}
$$

(4) $CR_{eac} = A_3B_3 + (CR_2)(A_3 + B_3)$

substituting (1), (2), and (3) into (4)

$$
\begin{aligned}
CR_{eac} = A_3B_3 &+ (A_2B_2)(A_3 + B_3) \\
&+ (A_1B_1)(A_2 + B_2)(A_3 + B_3) \\
&+ (A_0B_0)(A_1 + B_1)(A_2 + B_2)(A_3 + B_3)
\end{aligned}
$$

Notice the term, $(EAC)(A_0 + B_0)(A_1 + B_1)(A_2 + B_2)(A_3 + B_3)$, is omitted for an *EAC* cannot be obtained from itself.

C. Final sum operation

1. $S_0 = HA_0\overline{EAC} + \overline{HA}_0EAC$
2. $S_1 = HA_1\overline{CR}_0 + \overline{HA}_1CR_0$
3. $S_2 = HA_2\overline{CR}_1 + \overline{HA}_2CR_1$
4. $S_3 = HA_3\overline{CR}_2 + \overline{HA}_3CR_2$

General

A. Half-add operation

(1) $HA_n = A_n\overline{B}_n + \overline{A}_nB_n$

B. Carry operation

(1) $CR_n = A_nB_n + (CR_{n-1})\,(A_n + B_n)$

(2) $EAC = A_NB_N + (CR_{N-1})\,(A_N + B_N)$
$= CR_{0-1}$

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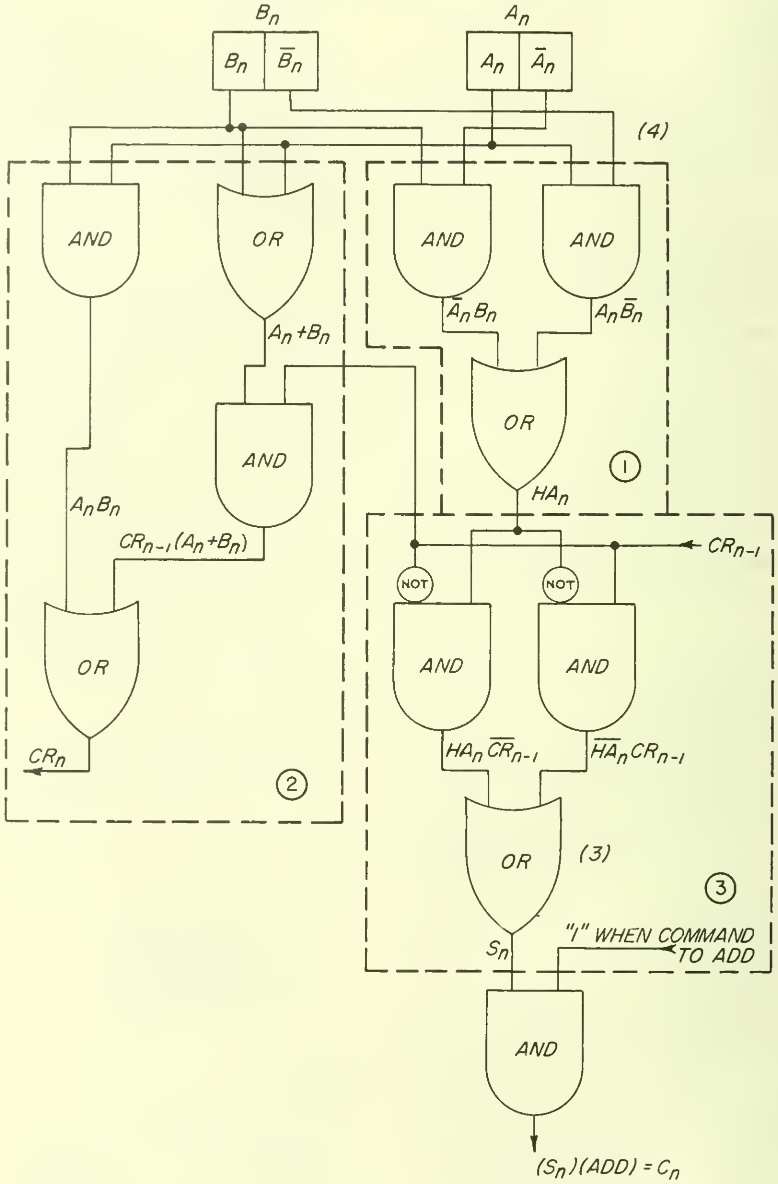

Figure 4-6.—Logic diagram mechanization of $A+B$ to C (an adder).

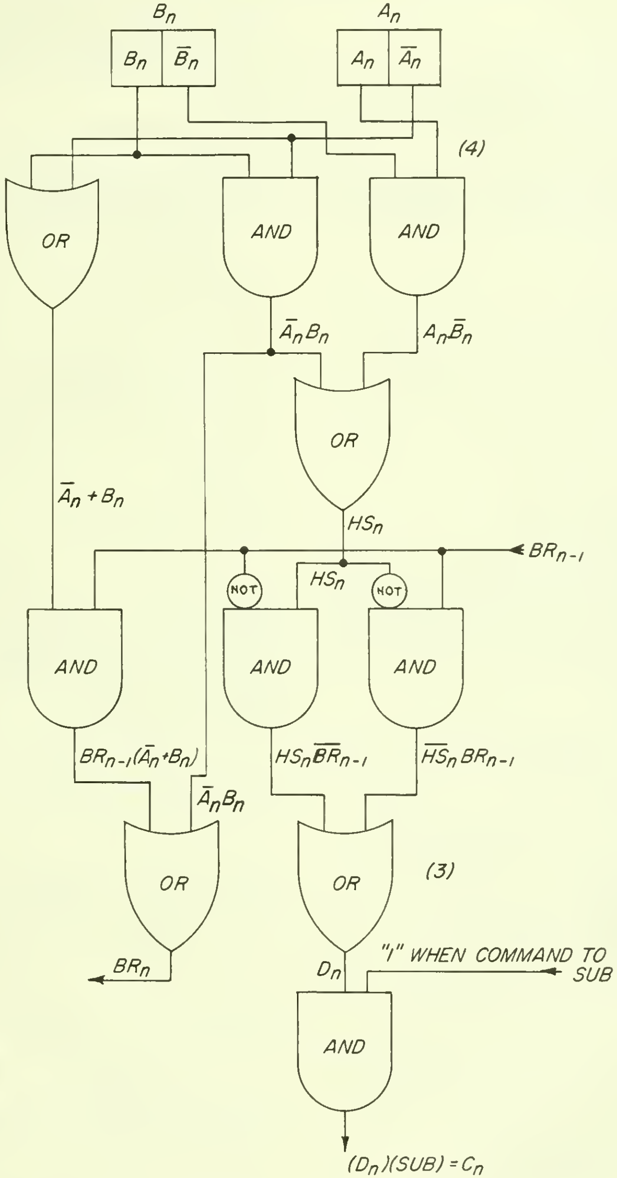

Figure 4-7.—Logic diagram mechanization of *A—B* to C (a subtractor).

C. Final sum operation

(1) $S_n = HA_n\overline{CR}_{n-1} + \overline{HA}_nCR_{n-1}$

where n is equal to the 2^n level and N is the highest level of the register (the most significant place level).

Operation: $A-B$ to C

Truth table for half-subtract/borrow process

A	0	0	1	1
B	0	1	0	1
Half-sub	0	1	1	0
Borrow	0	1	0	0

Sample problem:

$$A = 1_3\,0_2\,1_1\,1_0 = \quad -4$$
$$\underline{B = 1_3\,1_2\,0_1\,1_0 = \quad -2}$$
$$HS = 0_3\,1_2\,1_1\,0_0$$
$$\underline{BR = 1_2\,0_1\,1_0\,1_{eab}}$$
$$DIF = 1_3\,1_2\,0_1\,1_0 = \quad -2$$

where BR_2 is the first borrow generated and EAB is a result of the BR_2 not being satisfied at the third level, and the BR_0 is a result of the EAB not being satisfied at the zero level.

Boolean Expressions:

General

A. Half-subtract operation

(1) $HS_n = A_n\overline{B}_n + \overline{A}_nB_n$

B. Borrow operation

(1) $BR_n = \overline{A}_nB_n + (BR_{n-1})\,(\overline{A_n + B_n})$

(2) $\mathrm{EAB} = A_NB_N + (BR_{N-1})\,(\overline{A_N + B_N})$
$= BR_{0-1}$

C. Difference operation

(1) $D_n = HS_n\overline{BR}_{n-1} + \overline{HS}_nBR_{n-1}$

Once the Boolean expressions are derived from the sample problem and truth table, we can express them by logic diagram (digital graphic symbols) mechanization. This is done by following the foregoing rules.

(1) Given: A Boolean function i.e., $f(AB) = A+B$, $f(AB) = A-B$

(2) Attempt to simplify function
  a. in an *AND* to *OR*, or an *OR* to *AND* form.
  b. perform necessary negations if *NAND*, or *NOR* logic is to be used.

(3) Start at the final output, employing either an *AND*, or an *OR* for the final output logic diagram. (Some cases may require or dictate a *NOT* diagram for the simplest form.)

(4) Establish final diagram so as to employ only single line inputs for each variable, except in those cases where the variables originate from a source which has both variables available.

(5) Connect all diagrams of functions to complete the desired network.

(Refer to figures 4–6 and 4–7 for rules (3) and (4) above. These figures are the final logic diagram mechanizations.)

Notice on figure 4–6 the enclosures made by dotted lines. The logic diagram mechanizations enclosed represent the three general expressions of the additive process.

(1) $HA_n = A_n\overline{B}_n + \overline{A}_nB_n$

The inputs are A_n, $\overline{B}_n$, $\overline{A}_n$, and B_n with the HA_n as the output.

(2) $CR_n = A_nB_n + (CR_{n-1})\,(A_n + B_r)$

The inputs are A_n, B_n, and CR_{n-1}, with the CR_n as the output.

(3) $S_n = HA_n\overline{CR}_{n-1} + \overline{HA}_nCR_{n-1}$

The inputs are HA_n and CR_{n-1} with the S_n as the output.

CHAPTER 5

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

5-1. Introduction

Beginning with this chapter and through the remainder of the book, we will be dealing with the mathematics known as the calculus; both differential and integral calculus. The meanings of these terms will become evident as we proceed.

It is recommended that the student work all examples and exercises as many points not mentioned in the main body of the text are introduced in the examples and exercises.

5-2. Functional Meaning and Notation

Suppose we see an equation $y=x$. It should be readily apparent from our previous mathematics courses that y is a function of x or, reworded, assign a value to x and y is determined. We say that x is the INDEPENDENT variable and y is the DEPENDENT variable. Any mathematical changes introduced by some constant will not alter the fact that y is a function of x alone. The notation usually used for this relationship is $y=f(x)$ and in the equation $y=x$, $f(x)=x$.

In this case the choice was arbitrary for we could have just as easily chosen x as the dependent variable and y as the independent variable. In any particular physical problem however, the choice is not normally arbitrary. For example, in a d-c circuit with a given battery, the current that flows is dependent on the resistance we insert. In this case the resistance, R, would be the independent variable and the current, i, would be the dependent variable since its value varies as we vary R.

If the expression is in fact solved for the dependent variable, y, in terms of the independent variable, x, we say that y is an explicit function of x.

Example 1.

$$y=\frac{x^2}{3}+2x+1$$

This is an EXPLICIT function of the dependent variable y in terms of the independent variable x.

If the expression is not in fact solved for the dependent variable, y, in terms of the independent variable, x, we say that y is an IMPLICIT function of x.

Example 2.

$$3xy+x^2y^2=16$$

This is an implicit function of y in terms of x. The selection of which is the dependent variable and which is the independent variable is again arbitrary until we know the problem to be solved.

Example 3.

Given $y=\frac{x^2}{3}+2x+3$

What values will the dependent variable acquire when the independent variable is 2, 3, 10?

Answer. 8 1/3, 12, 56 1/3

Example 4.

Given $4xy+2y=3$

Is this an implicit or an explicit function of y? When the independent variable, x, is assigned the value 1, what value will the dependent variable have? Answer. Implicit, 1/2

5-3. Division by Zero is Not Defined

We may recall when learning the fundamentals of division that it was possible to "prove" our work by multiplying the divisor by the quotient to obtain the dividend providing our work was done correctly.

Thus, we may verify $24\div 3=8$ by forming,

$$\text{divisor}\times\text{quotient}=\text{dividend}$$
$$3\times 8=24$$

and hence the proof.

In a like manner let us attempt to verify the following;

$$y \div 0 = z$$
$$\text{divisor}\times\text{quotient} = \text{dividend}$$
$$0\times z = 0$$

Obviously it doesn't check. Note also that it will check only in the case when the dividend, y, is zero. But in this event, the quotient, z, could take any value for 0 times anything $=0$.

Therefore, the operation of dividing by zero is meaningless and we say that it is an undefined mathematical operation. WE MAY NEVER DIVIDE BY ZERO, and care must be used to avoid such an operation.

Example 5.

What value does the function $y=\dfrac{x^2+1}{x^2-2x+1}$ attain when $x=1$?

(A) When $x=1$, the function is $y=\dfrac{1+1}{1-2+1}=\dfrac{2}{0}$

Thus, the function is UNDEFINED at $x=1$.

5–4. Limits of Variables and Functions

The idea of calculating limits of variables and functions will be used directly and indirectly throughout the remainder of the book. We will attempt to deal mainly with the working concepts rather than the theoretical concepts.

Stated formally the concept of the limit of a variable is as follows: the variable V is said to approach the constant L as a limit when the successive values of V are such that the numerical value of the difference $(V-L)$ becomes and remains less than any preassigned positive number.

Less formally we may say that suppose $V=f(x)$ and as we assign values to x nearer and nearer to some value, a, V approaches nearer and nearer to L. Therefore the limit of V is L as x approaches a.

We note this as $\lim\limits_{x\to a} V=L$

An example will help to clarify this point.

Example 6.

$$y=f(x) \text{ and } f(x)=4/\sqrt{x}$$
$$\therefore y=4/\sqrt{x}$$

The three dots ∴ mean "therefore." In this example we are interested in only positive values of x and y, that is, $x>0$ and $y>0$ (fig. 5–1).

Notice as we choose values for x closer and closer to 4, y assumes a value nearer and nearer to 2. We say the limit of y or $f(x)$ as x approaches 4 is 2. In notation form

$$\lim_{x\to 4} f(x)=\lim_{x\to 4} y=\frac{4}{\sqrt{x}}=2.$$

(Remember we are interested only in the positive root in this case.)

If we approach 4 from more positive values, that is from the right-hand side of 4, we call this the right-hand limit and note it as

$$\lim_{x\to 4+} y=2$$

If we approach 4 from less positive values, that is from the left-hand side of 4, we call this the left hand limit and note it as

$$\lim_{x\to 4-} y=2$$

It will almost always be true that with the functions in which we are interested, the left- and right-hand limits will be equal.

With this background we are prepared to examine several theorems concerning limits. Let us consider three different functions of x; $f(x)=y$, $g(x)=w$, and $\Phi(x)=6V$. Suppose that the following limits are valid for our chosen functions of x;

$$\lim_{x\to L} f(x)=\lim_{x\to L} y=A,$$

$$\lim_{x\to L} g(x)=\lim_{x\to L} w=B,$$

and

$$\lim_{x\to L} \Phi(x)=\lim_{x\to L} v=C$$

Considering these three functions we have:

Theorem 1. The sum of the limits is equal to the limit of the sums.

Sum of the limits = limit of the sums

$$\lim_{x\to L} y+\lim_{x\to L} w+\lim_{x\to L} v=\lim_{x\to L}(y+w+v)=A+B+C$$

Note that this theorem will also hold for subtractions in any order since the subtraction process is merely the addition of negative quantities.

Theorem 2. The product of the limits is equal to the limit of the products.

$$\left[\lim_{x\to L} y\right]\left[\lim_{x\to L} w\right]\left[\lim_{x\to L} v\right]=\lim_{x\to L}(ywv)=ABC$$

Theorem 3. The quotient of the limits is equal to the limit of the quotients.

$$\frac{\left[\lim_{x\to L} y\right]\left[\lim_{x\to L} w\right]}{\left[\lim_{x\to L} V\right]}=\lim\left(\frac{yw}{v}\right)=\frac{AB}{C}$$

provided that $\lim_{x\to L} v\neq 0$ since division by zero is undefined.

5–5. The Meaning of Infinity

Let us again look at figure 5–1. What value does y approach as x approaches zero. (We must be careful not to say as x becomes equal to zero for we may never divide by zero.) We can see from the graph that y becomes positive and increasingly large.

When a variable becomes and remains greater than any number we may assign, we say that variable is approaching INFINITY. If the variable is positive and increasing in this manner, it is approaching $+\infty$. If the variable is negative and decreasing, it is approaching $-\infty$. In the event we are not concerned about the sign of the function, we say it is approaching infinity without specifying which direction.

A function that is approaching infinity in either direction is termed UNBOUNDED.

We note these cases as $\lim y=\infty$, $\lim y=+\infty$, or $\lim y=-\infty$.

In our particular equation of $y=\frac{4}{\sqrt{x}}$, no matter what number we may arbitrarily assign y, we may exceed this number by choosing x nearer and nearer to zero. (But never equal to zero.) We may write then $\lim_{x\to 0}\frac{4}{\sqrt{x}}=+\infty$. We choose $+\infty$ since we have limited our values to positive values of x and y.

Operations with infinity may be summarized as follows:

(a) Any number multiplied by infinity is infinity. $C\,\infty=\infty$ where C is a constant

(b) Any number divided by infinity is zero. $C/\infty=0$

(c) Any number added to infinity is infinity. $C+\infty=\infty$

Several example problems on finding limits of functions follow:

Example 7.

(a) $\lim_{x\to 2}(4x^2+3x)=22$

(b) $\lim_{x\to 0}\frac{3x^2+2}{x^2+2x+1}=2$

(c) $\lim_{x\to 2}\frac{x-2}{x^2-2}=\frac{0}{2}=0$

In the above problems we allow x to approach the assigned values, but never actually reach them. In these simple examples however, the result may be attained by actual substitution. In more complex cases this will not work.

5–6. Procedures for Calculating Limits of Indeterminant Forms

Indeterminant forms occur in some cases when we attempt to find the limit of some function by substitution as mentioned in section 5–5. Forms which we shall term as indeterminant are:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty\times 0,\ 0^0,\ \infty^0,\ 1^\infty,\ \text{and}\ \infty-\infty.$$

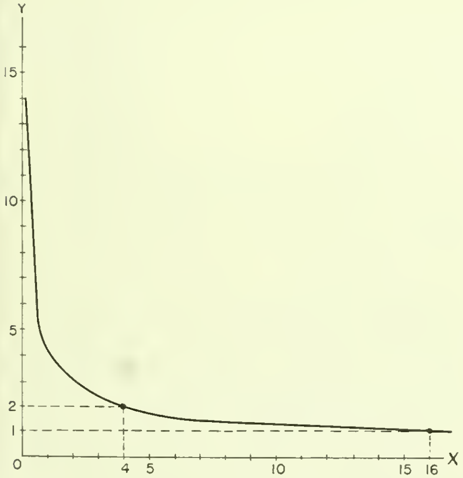

Figure 5–1.—Graph of $y=\frac{4}{\sqrt{x}}$.

Powerful theorems are available for handling these forms, but they are beyond the intention of this text. If further information is desired, refer to any advanced calculus text. However, we will later examine the form $\frac{0}{0}$.

Simple algebraic manipulations will enable us to deal with most cases in which we will be interested, and experience will dictate the procedure to follow. The solutions to the following problems will be of assistance.

Example 8.

Verify the following solutions.

$$\text{(a)}\ \lim_{x\to\infty}\frac{x^3+2x^2+x-7}{3x^3+x^2+4x+1}=\frac{1}{3}$$

(A) We first see that direct substitution will yield the indeterminant form $\frac{\infty}{\infty}$ since ∞ added (or subtracted) to any number is ∞. We must proceed in another manner.

(B) Divide both numerator and denominator by the highest power of x that occurs in either. Here x^3 occurs in both so we divide by x^3.

$$\lim_{x\to\infty}\frac{\frac{x^3}{x^3}+\frac{2x^2}{x^3}+\frac{x}{x^3}-\frac{7}{x^3}}{\frac{3x^3}{x^3}+\frac{x^2}{x^3}+\frac{4x}{x^3}+\frac{1}{x^3}}$$

$$=\lim_{x\to\infty}\frac{1+\frac{2}{x}+\frac{1}{x^2}-\frac{7}{x^3}}{3+\frac{1}{x}+\frac{4}{x^2}-\frac{1}{x^3}}$$

(C) Recalling that any number divided by ∞ is zero we have

$$\lim_{x\to\infty}\frac{1+\frac{2}{x}+\frac{1}{x^2}-\frac{7}{x^3}}{3+\frac{1}{x}+\frac{4}{x^2}+\frac{1}{x^3}}=\frac{1}{3}$$

$$\text{(b)}\ \lim_{t\to 0}\frac{7t^2+3t}{t^2-2t}=-\frac{3}{2}$$

(A) In this case direct substitution yields the indeterminant form $\frac{0}{0}$ and again we must look for a different procedure.

(B) Factor the common factor, t, from both the numerator and denominator

$$\lim_{t\to 0}\frac{t(7t+3)}{t(t-2)}$$

(C) Cancel a t from each common factor in the numerator and denominator yielding

$$\lim_{t\to 0}\frac{7t+3}{t-2}=-\frac{3}{2}$$

$$\text{(c)}\ \lim_{V\to 2}\frac{V^2-4}{V-2}=4$$

(A) Direct substitution again yields $\frac{0}{0}$

(B) Factor the numerator and cancel

$$\lim_{V\to 2}\frac{(V-2)(V+2)}{V-2}=\lim_{V\to 2}V+2$$

(C) $\lim_{V\to 2} V+2=4$

The following exercise is left for the student to verify.

Exercise 5–1.

Find the limit of each of the following functions.

1. $\lim_{x\to 1}\frac{3x^2-x}{2}$
2. $\lim_{x\to a}\frac{x^2-a}{3a}$
3. $\lim_{x\to t}\frac{t^2-2x+1}{x-t}$
4. $\lim_{t\to\infty}\frac{t^2+2t+1}{2t^2+3t-4}$
5. $\lim_{y\to 0}\frac{y^3-8}{y-2}$
6. $\lim_{x\to 4}\frac{x^2-16}{x-4}$
7. In a simple RC series circuit the voltage at any time t is given by $V=V_m(1-e^{-\frac{t}{RC}})$. What is the value of V as $t\to 0$; $t\to\infty$? Answer. $0, V_m$

5–7. Infinitesimals

Any variable that approaches zero as a limit is termed an INFINITESIMAL.

Refer to section 5–4 where we listed the formal definition of a limit. Notice that the difference between a variable and its limit is an infinitesimal, for we state that the difference between the variable, V, and its limit, L, becomes and remains less than any preassigned positive number. In other words the difference between V and L approaches zero.

Let us see what happens when we have the product of two variables, η and ϵ (fig. 5–2). The initial value of η is 8 and ϵ is 1. We first allow ϵ to approach zero, that is, we make ϵ an infinitesimal.

What happens to the product $\eta\epsilon$ if $\eta = 8$ and does not change as $\epsilon \to 0$? In the vertical columns labeled A through H, we see that the values of $\eta\epsilon$ range from 8 in column A to $\frac{1}{16}$ in column H. If we continued as $\epsilon \to 0$, the product $\eta\epsilon$ would approach zero also.

Now suppose we say that $\eta \to 0$ along with ϵ and at the same rate as ϵ. By the same rate we mean in this example that if the value for ϵ is read in column C, the value for η is read in row C. The values for the product $\eta\epsilon$ now range from 8 in column A and row A to $\frac{1}{2048}$ in column H and row H. This is much less than $\frac{1}{16}$ which was the value of $\eta\epsilon$ when η was fixed at 8 and the value for ϵ was read in column H as $\epsilon \to 0$.

This example is meant to illustrate the fact that THE PRODUCT of two infinitesimals will approach zero faster than the product of a single infinitesimal and any number. In a like manner we could show that the product of three infinitesimals would reach zero even faster than two, etc.

The product of two infinitesimals we call a SECOND ORDER infinitesimal, and the product of three infinitesimals we call a THIRD ORDER infinitesimal, etc.

5-8. Continuity

A function $f(x)$ is CONTINUOUS AT A POINT P if the limit of $f(x)$ at the point P is equal to the function evaluated at the point P, $[f(P)]$. If $\lim_{x \to P} f(x) = f(P)$, the function is continuous. If a function is not continuous, it is said to be discontinuous.

		A	B	C	D	E	F	G	H
	$\eta \backslash \varepsilon$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128} \to 0$
A	8	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
B	4	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
C	2	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
D	1	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
E	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
F	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$
G	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$
H	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$

$\downarrow 0$

Figure 5–2.—Product of 7ε

When we speak of a function being continuous, we imply that it is continuous over the range of values with which we are interested. We shall only test for continuity at a point and thus the entire range of values should be investigated point by point. In most instances of concern to us the question of continuity will be apparent.

Example 9.

Investigate the following functions for continuity at the indicated points.

$$\text{(a) } f(x)=x^2+1 \text{ at } x=4$$

(A) $\lim_{x\to 4} x^2+1=17$ and $f(4)=16+1=17$

(B) Since $\lim_{x\to 4} f(x)=f(4)$, the function x^2+1 is CONTINUOUS at $x=4$.

$$\text{(b) } f(x)=\frac{x^2-4}{x-2} \text{ at } x=2$$

$$\text{(A) } \lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}=\lim_{x\to 2} x+2=4$$

and $f(2)$ has the indeterminant form $\frac{0}{0}$ and hence $f(x)$ is undefined at $x=2$.

(B) Since $\lim_{x\to 2} f(x)$ IS NOT EQUAL to $f(2)$, the function is DISCONTINUOUS at $x=2$.

$$\text{(c) } f(x)=\frac{1}{x-1} \text{ at } x=1$$

(A) $\lim_{x\to 1}\frac{1}{x-1}=\infty$. The limit does not exist and we say the limit APPROACHES infinity (not EQUALS infinity).

(B) The function does not exist at $x=1$ so that the function is DISCONTINUOUS at $x=1$.

A graph of this function is given in figure 5–3.

5–9. Increments

An increment of a variable is the change in value of that variable. If x were to change from 2 to 5, we

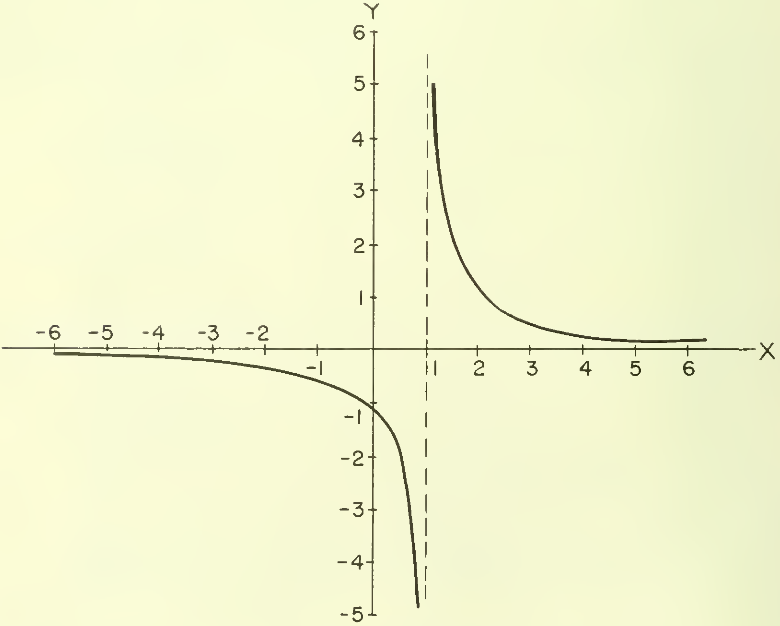

Figure 5–3.—Plot of $y=f(x)=\frac{1}{x-1}$.

would find the increment of x (written Δx and stated "delta x") by subtracting the initial value, 2, from the final value, 5. That is $\Delta x = x_f - x_i = 5-2 = 3$. The increment of a variable is ALWAYS the final value minus the initial value. If the increment is positive, the variable is increasing in value and if the increment is negative, the variable is decreasing in value.

Example 10.

Find Δy (increment of y) in the following function with Δx as given.

$$y = f(x) = 2x^2 \text{ and } x_i = 5, x_f = 7$$

Two methods will be shown.

Method 1. Calculate y_i and y_f from the corresponding values of x and then find Δy from $y_f - y_i$.

$$y_i = 2x_i^2 = (2)(5)^2 = 50$$

$$y_f = 2x_f^2 = (2)(7)^2 = 98$$

$$\Delta y = y_f - y_i = 98 - 50 = 48$$

Method 2. When x changes from 5 to 7, we assign $x=5$ and $\Delta x = 2$ and y will also change. We indicate this as

$$y + \Delta y = f(x+\Delta x), \text{ but } y = f(x)$$

$$\therefore \quad f(x) + \Delta y = f(x+\Delta x)$$

$$\Delta y = f(x+\Delta x) - f(x)$$

Now since $x + \Delta x = 7$ and $x = 5$, $\Delta y = f(7) - f(5)$ and since $f(x) = 2x^2$, $f(7) = 2(7^2) = 98$, $f(5) = 2(5^2) = 50$ and $\Delta y = 98 - 50 = 48$

5-10. Average and Instantaneous Rates of Change

A natural extension of the increment concept would be to ask, "How rapidly does the dependent variable change with respect to the independent variable?"

Suppose that we traveled the road between two towns, a distance of 30 miles. Let us say we did this in about one hour. We could say that we AVERAGED 30 mph for the trip. In formula form our average velocity, $\bar{v}$ (read "v bar") is equal to the distance traveled divided by the overall time elapsed, i.e., $\bar{v} = \dfrac{\text{total Distance}}{\text{time elapsed}}$.

Here our increment of distance, ΔD, is 30 miles and our increment in time, Δt, is 1 hour. We have then $\bar{v} = \dfrac{\Delta D}{\Delta t}$ which is our AVERAGE VELOCITY or the AVERAGE RATE of change of distance with respect to time.

Note in particular that we have made no mention of how our speed varied at any instant since it probably fluctuated from 60 mph on the freeways to zero mph at the traffic lights. We have only indicated our AVERAGE RATE.

In a similar manner we may calculate the average rate of change of y with respect to x where y is some function of x.

We already know (1) $y = f(x)$ and (2) $y + \Delta y = f(x + \Delta x)$.

Subtracting (1) from (2) we get,

$$y + \Delta y = f(x+\Delta x) \qquad (2)$$

$$\underline{-[y = f(x)]} \qquad (1)$$

$\Delta y = f(x+\Delta x) - f(x)$ giving us the familiar expression for Δy. However, we must divide through by Δx to give the AVERAGE change of y with respect to the change of x.

$$\therefore \qquad \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Now we return to our cross-country journey. We could calculate our INSTANTANEOUS rate of change of distance with respect to time by choosing some segment of our total increment of time (choose Δt_1 where $\Delta t = \text{total time} = \Delta t_1 \Delta t_2 + \ldots + \Delta t_n$) small enough so that our velocity would be nearly constant during this subincrement of time. In other words, choose $\Delta t_1 \to 0$! This would enable us to calculate our instantaneous velocity during the time Δt_1.

$$v_1 = \text{instantaneous velocity during } \Delta t_1$$
$$= \lim_{\Delta t_1 \to 0} \frac{\Delta D}{\Delta t_1}$$

This may again be extended to calculate the INSTANTANEOUS rate of change of y with respect to x by allowing $\Delta x \to 0$. The instantaneous rate of change is called the DERIVATIVE of y with respect to x. In notation form we have:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Symbols used to indicate the derivative include $\frac{dy}{dx}$, $f'(x)$, y', $D(x)$, and others.

We will limit our use to $\frac{dy}{dx}$ and y'.

Since we define the derivative,
$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$, it follows that in order for the derivative to exist, the defining limit must exist. The operation of finding the derivative is called differentiating.

The general procedure or rule for finding the derivative of any function is given below. This general rule is sometimes called the "delta process."

(1) Assume $y=f(x)$ so that
(2) $y+\Delta y=f(x+\Delta x)$
(3) subtract 1 from 2: $\Delta y=f(x+\Delta x)-f(x)$
(4) divide by Δx: $\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$
(5) take the limit as $\Delta x \to 0$:
$$\frac{dy}{dx}=\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x \to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Example 5–11.

(a) Given the ohm's law relationship of $V = iR$ where V is the voltage, i is the variable current, and R is the constant resistance so that $V=f(i)$. Calculate the rate of change of voltage with respect to the current, i.e., $\frac{dV}{di}$.

Following the steps given in the general rule, we have:

(1) $V=iR$
(2) $V+\Delta V=(i+\Delta i)R$
(3) $\Delta V=(i+\Delta i)R-iR=\Delta iR$
(4) $\frac{\Delta V}{\Delta i}=\frac{\Delta iR}{\Delta i}=R$
(5) $\frac{dV}{di}=\lim_{\Delta i \to 0}\frac{\Delta V}{\Delta i}=R$

Therefore the derivative of the voltage with respect to the current is a constant equal to the resistance.

(b) Given $P=i^2R$ where P is the power given off as heat in the resistor R and i is the variable current passing through the resistor. Calculate the rate of change of the power with respect to the current, i.e., $\frac{dP}{di}$, when the current is 2 amperes.

(1) $P=i^2R$
(2) $P+\Delta P=(i+\Delta i)^2R=(i^2+2i\Delta i+\Delta i^2)R$
(3) $\Delta P=(2i\Delta i+\Delta i^2)R$
(4) $\frac{\Delta P}{\Delta i}=\frac{2i\Delta iR+\Delta i^2R}{\Delta i}=2iR+\Delta iR$
(5) $\frac{dP}{di}=\lim_{\Delta i \to 0}\frac{\Delta P}{\Delta i}=\lim_{\Delta i \to 0} 2iR+\Delta iR=2iR$

$\therefore \frac{dP}{di}=2iR$ so that when i is 2 amperes, the derivative $=4R$.

Exercise 5–2.

Using the procedure of the two previous examples, calculate the derivatives of the following functions.

1. $y=mx+b$
2. $y=2x^2+1$
3. $y=x-3x^2$
4. $y=4ax$
5. $y=\frac{1}{x^2}$
6. $y=\frac{x+1}{x}$

5–11. Theorems for Differentiating Algebraic Functions

So far in our work we have utilized the general form for differentiating functions. While important in understanding what principles are involved in the differentiating process, it is far too cumbersome for general practical use.

The following theorems must be committed to memory, BUT ONLY AFTER A THOROUGH UNDERSTANDING OF THEIR DEVELOPMENT HAS BEEN ATTAINED.

The formulas are first listed without explanation for ease of reference. Derivations using the general rule for differentiation follow u and v are to be considered as functions of x while c is a constant. (The words formula and theorem will be interchangeable in this text for the formula symbolically represents the theorem.)

1. $\frac{d(c)}{dx}=0$
2. $\frac{d(x^n)}{dx}=nx^{n-1}$
3. $\frac{d(u+v)}{dx}=\frac{du}{dx}+\frac{dv}{dx}$
4. $\frac{d(cv)}{dx}=c\frac{dv}{dx}$
5. $\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$
6. $\frac{d\left(\frac{u}{v}\right)}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

7. $\frac{d(u^n)}{dx}=nu^{n-1}\frac{du}{dx}$

8. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ where $y=f(u)$, $u=\phi(x)$

9. $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$ where $y=f(x)$

10. $\frac{d(cx)}{dx}=c$

Derivations of the above formulas

1. $y=c\ \frac{dy}{dx}=0$

Since y has the same value for all values of x, any change, Δx, in x does not affect y and $\Delta y=0$.

$$\frac{\Delta y}{\Delta x}=0, \text{ but } \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}=0$$

$$\therefore \frac{dc}{dx}=0$$

2. $y=x^n\ \frac{dy}{dx}=nx^{n-1}$

A. Allow x to vary by Δx;

$$\therefore y+\Delta y=(x+\Delta x)^n$$

B. Calculate Δy:

$$\Delta y=(x+\Delta x)^n-x^n$$

C. Divide by Δx and take the limit as $\Delta x\to 0$;

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{(x+\Delta x)^n-x^n}{\Delta x}$$

D. Expand $(x+\Delta x)^n$ by the binomial expansion;

$$(x+\Delta x)^n=x^n+nx^{n-1}\Delta x+\frac{n(n-1)}{2}x^{n-2}\Delta x^2+\ldots+\Delta x^n$$

E. Substitute in expression for $\frac{dy}{dx}$;

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{x^n+nx^{n-1}\Delta x+\frac{n(n-1)}{2}x^{n-2}\Delta x^2+\ldots+\Delta x^n-x^n}{\Delta x}$$

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\ nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}\Delta x+\ldots+\Delta x^{n-1}$$

$$\frac{dy}{dx}=nx^{n-1}$$

Example 5–12.

Find the derivative of $y=x^3$. Where $n=3$ and $n-1=2$

$$\therefore \frac{dy}{dx}=3x^2$$

3. $y=u+v$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dv}{dx}$$

A. Allowing x to assume an increment Δx will cause u to vary by Δu and v by Δv;

$$\therefore y+\Delta y=u+\Delta u+v+\Delta v$$

B. Calculate Δy;

$$\Delta y=u+\Delta u+v+\Delta v-(u+v)$$
$$\Delta y=\Delta u+\Delta v$$

C. Divide by Δx and take the limit as $\Delta x\to 0$:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}$$

so that $\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$

Example 5–13.

Find the derivative of $y=x^5+x+7$. Here $u=x^5$, $v=x$ and $c=7$.

By formulas 1, 2, and 3:

$$\frac{dy}{dx}=5x^4+1$$

4. $y=cv$

$$\frac{dy}{dx}=c\frac{dv}{dx}$$

A. Allowing x to vary by Δx will cause a change Δy in y and Δv in v, but no change in c.

$$y+\Delta y=c(v+\Delta v)$$

B. Calculate Δy;

$$\Delta y=c(v+\Delta v)-cv$$

C. Divide by Δx and take the limit as $\Delta x\to 0$;

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}c\frac{\Delta v}{\Delta x}$$

$$\frac{dy}{dx}=c\frac{dv}{dx}$$

Example 5–14.

Find the derivative of $y=5x^5$.

Here $c=5$ and $v=x^5$;

$$\therefore \frac{dy}{dx}=5(5x^4)=25x^4$$

5. $y=uv \qquad \dfrac{dy}{dx}=v\dfrac{du}{dx}+u\dfrac{dv}{dx}$

A. Allowing x to vary by Δx will cause a change Δy in y, Δu in u, and Δv in v;

$$y+\Delta y=(u+\Delta u)(v+\Delta v)$$

B. Calculate Δy;

$$\Delta y=uv+v\Delta u+u\Delta v+\Delta u\Delta v-uv$$

C. Divide by Δx and take the limit as $\Delta x\rightarrow 0$:

$$\frac{dy}{dx}=\lim_{\Delta x\rightarrow 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\rightarrow 0} v\frac{\Delta u}{\Delta x}+u\frac{\Delta v}{\Delta x}+\Delta u\frac{\Delta v}{\Delta x}$$

so that $\dfrac{dy}{dx}=v\dfrac{du}{dx}+u\dfrac{dv}{dx}$ since $\lim_{\Delta x\rightarrow 0}\dfrac{\Delta u\Delta v}{\Delta x}=0.$

Notice in the expression $\dfrac{\Delta u\Delta v}{\Delta x}$ that the numerator is a second order infinitesimal and the denominator is a first order infinitesimal. For that reason the numerator will approach zero more rapidly than the denominator (see section 5–8).

Example 5–15.

Find the derivative of $y=(x^2+1)(3x^2+2)$. We have $u=(x^2+1)$, $v=(3x^2+2)$, and $\dfrac{du}{dx}=2x$, $\dfrac{dv}{dx}=6x$.

$$\therefore \frac{dy}{dx}=(3x^2+2)2x+(x^2+1)6x=2x(3x^2+2+3x^2+3)$$

6. $y=\dfrac{u}{v} \qquad \dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

A. Allow x to vary by Δx:

$$y+\Delta y=\frac{u+\Delta u}{v+\Delta v}$$

B. Calculate Δy:

$$\Delta y=\frac{u+\Delta u}{v+\Delta v}-\frac{u}{v}=\frac{uv+v\Delta u-uv-u\Delta v}{v(v+\Delta v)}$$

$$\Delta y=\frac{v\Delta u-u\Delta v}{v(v+\Delta v)}$$

C. Divide by Δx and take the limit as $\Delta x\rightarrow 0$.

$$\frac{dy}{dx}=\lim_{\Delta x\rightarrow 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\rightarrow 0}\frac{v\dfrac{\Delta u}{\Delta x}-u\dfrac{\Delta v}{\Delta x}}{v(v+\Delta v)}=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$$

since $\Delta v\rightarrow 0$ as $\Delta x\rightarrow 0$.

Example 5–16.

Find the derivative of $y=\dfrac{x-3}{x^2+2x-1}$

$u=x-3$, $v=x^2+2x-1$, and $\dfrac{du}{dx}=1$, $\dfrac{dv}{dx}=2x+2$

$$\therefore \frac{dy}{dx}=\frac{(x^2+2x-1)(1)-(x-3)(2x+2)}{(x^2+2x-1)^2}$$

$$\frac{dy}{dx}=\frac{x^2+2x-1-2x^2+4x+6}{(x^2+2x-1)^2}=\frac{5+6x-x^2}{(x^2+2x-1)^2}$$

7. $y=u^n \qquad \dfrac{dy}{dx}=nu^{n-1}\dfrac{du}{dx}$

A. Allow x to vary by Δx; $y+\Delta y=(u+\Delta u)^n$

B. Calculate Δy; $\Delta y=(u+\Delta u)^n-u^n$

C. Divide by Δx and take the limit as $\Delta x\rightarrow 0$.

$$\frac{dy}{dx}=\lim_{\Delta x\rightarrow 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\rightarrow 0}\frac{(u+\Delta u)^n-u^n}{\Delta x}$$

D. Expand $(u+\Delta u)^n$ by the binomial expansion;

$$(u+\Delta u)^n=u^n+nu^{n-1}\Delta u+\frac{n(n-1)}{2}u^{n-2}\Delta u^2+\ldots+\Delta u^n$$

E. Substitute the expansion in D into the expression for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx}=\lim_{\Delta x\rightarrow 0}\frac{u^n+nu^{n-1}\Delta u+\dfrac{n(n-1)}{2}u^{n-2}\Delta u^2+\ldots+\Delta u^n-u^n}{\Delta x}$$

$$\frac{dy}{dx}=\lim_{\Delta x\rightarrow 0}\left[nu^{n-1}\frac{\Delta u}{\Delta x}+\frac{n(n-1)}{2}u^{n-1}\Delta u\frac{\Delta u}{\Delta x}+\ldots+\frac{\Delta u^n}{\Delta x}\right]$$

$\dfrac{dy}{dx}=nu^{n-1}\dfrac{du}{dx}$ as all other terms approach zero in the limit as $\Delta x\rightarrow 0$.

Example 5–17.

Find the derivative of

$y=(3x^2+4x+1)^3$. $u=3x^2+4x+1$ and $n=3$.

$$\therefore \frac{dy}{dx}=3(3x^2+4x+1)^2\,(6x+4)$$

$$\frac{dy}{dx}=(18x+12)\,(3x^2+4x+1)^2$$

8. Assume we have y as a function of u, i.e., $y=f(u)$ and u as a function of x, i.e., $u=\Phi(x)$. We say that y is then a function of x through the function u.

We wish to prove that:

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$$

A. Allowing x to vary by Δx will in turn produce a change Δu in u and Δy in y:
$y+\Delta y=f(u+\Delta u) \quad u+\Delta u=\Phi(x+\Delta x)$

B. Calculate Δy and Δu;
$\Delta y=f(u+\Delta u)-f(u) \quad \Delta u=\Phi(x+\Delta x)-\Phi(x)$

C. Form the ratios $\frac{\Delta y}{\Delta u}$ and $\frac{\Delta u}{\Delta x}$;

$$\frac{\Delta y}{\Delta u}=\frac{f(u+\Delta u)-f(u)}{\Delta u}, \quad \frac{\Delta u}{\Delta x}=\frac{\Phi(x+\Delta x)-\Phi(x)}{\Delta x}$$

D. It is readily apparent that $\frac{\Delta y}{\Delta x}=\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$.

You may better grasp this by mentally canceling the Δu's on the right-hand side, but we don't in fact do this.

Now substitute the equivalents in part (C) in the expression for $\frac{\Delta y}{\Delta x}$;

$$\frac{\Delta y}{\Delta x}=\frac{f(u+\Delta u)-f(u)}{\Delta u}\cdot\frac{\Phi(x+\Delta x)-\Phi(x)}{\Delta x}$$

E. Take the limit as $\Delta x\to 0$ which in turn will cause $\Delta u\to 0$;

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

$$=\lim_{\Delta u\to 0}\frac{f(u+\Delta u)-f(u)}{\Delta u}\cdot\lim_{\Delta x\to 0}\frac{\Phi(x+\Delta x)-\Phi(u)}{\Delta x}$$

The first fraction on the right side of the equation is the definition for $\frac{dy}{du}$ and the second is $\frac{du}{dx}$.

$$\therefore \text{ We have } \frac{dy}{dx}=\frac{dy}{du}\,\frac{du}{dx}$$

Example 5–18.

Find the derivative of $y=u^3$ with respect to x when $u=3x^2+4x+1$
See example 5–16.

$$\frac{dy}{du}=\frac{d(u^3)}{du}=3u^2 \text{ and } \frac{du}{dx}=\frac{d(3x^2+4x+1)}{dx}=6x+4$$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=(3u^2)\,(6x+4)$$

$$=3(3x^2+4x+1)^2(6x+4)$$

$$\frac{dy}{dx}=(18x+12)\,(3x^2+4x+1)^2$$

Notice the results of examples 5–17 and 5–18 check as indeed they must. In many cases simple substitution of $\phi(x)$ for $f(u)$ is not so easily handled and hence the need for the important formula 8.

9. $y=f(x)$ and $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$

Formula 9 is useful when we deal with inverse functions. Suppose $y=f(x)$. It must be equally true that we may solve this same equation for x yielding $x=g(y)$. $y=f(x)$ and $x=g(y)$ are then called inverse functions. In particular, $y=f(x)$ is termed the direct function and $x=g(y)$ is termed the inverse function.

10. $y=cx \qquad \frac{dy}{dx}=c$

A. Allow x to vary by Δx; $y+\Delta y=c\,(x+\Delta x)$
B. Calculate Δy; $\Delta y=cx+c\Delta x-cx=c\Delta x$
C. Divide by Δx and take the limit as $\Delta x\to 0$

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}=c$$

Example 5–19.

Find the inverse function of the given direct function. The inverse function is found by transposing the equation and solving for the independent variables.

Direct function	Inverse function
$y=x^2-6$	$x=\pm\sqrt{y+6}$
$y=\tan x$	$x=\arctan y$
$y=\sqrt{x}$	$x=y^2$
$y=e^x$	$x=\ln y$

We proceed now to derive the relationship for $\frac{dy}{dx}$ as given in 9.

A. Given the inverse functions $y=f(x)$ and $x=\phi(y)$ and allow x and y to vary by Δx and Δy;

$\therefore y+\Delta y=f(x+\Delta x)$ and $x+\Delta x=\phi(y+\Delta y)$

B. Calculate Δy and Δx;

$\Delta y=f(x+\Delta x)-f(x) \qquad \Delta x=\Phi(y+\Delta y)-\Phi(y)$

C. Form the ratios $\frac{\Delta y}{\Delta x}$ and $\frac{\Delta x}{\Delta y}$;

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x},\ \frac{\Delta x}{\Delta y}=\frac{\Phi(y+\Delta y)-\Phi(y)}{\Delta y}$$

D. Form the products of the ratios $\frac{\Delta y}{\Delta x}$ and $\frac{\Delta x}{\Delta y}$;

$$\frac{\Delta y}{\Delta x}\cdot\frac{\Delta x}{\Delta y}=1 \quad \text{or} \quad \frac{\Delta y}{\Delta x}=\frac{1}{\frac{\Delta x}{\Delta y}}$$

E. From part C we take the limits of the ratios as follows;

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{dy}{dx}$$

$$\lim_{\Delta y\to 0}\frac{\Delta x}{\Delta y}=\lim_{\Delta y\to 0}\frac{\Phi(y+\Delta y)-\Phi(y)}{\Delta y}=\frac{dx}{dy}$$

F. Taking the limit of the relationships in D;

$$\frac{dy}{dx}\cdot\frac{dx}{dy}=1 \text{ or } \frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$$

Example 5–20.

Verify using formula 9 $\frac{dy}{dx}$ for the function $y=4x^2$.

By direct differentiation using formula 2 we get $\frac{dy}{dx}=8x$. In order to verify with formula 9 we first solve for x; $x^2=\frac{y}{4}$ and $x=\frac{1}{2}\,y^{1/2}$ (choosing the positive root only).

We form $\frac{dx}{dy}$ by applying formula 2:

$$\frac{dx}{dy}=\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\,y^{(1/2\,-1)}=\frac{1}{4y^{1/2}}$$

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\frac{1}{4y^{1/2}}}=4y^{1/2}=4(2x)=8x$$

This verifies our two procedures.

5–12. Implicit Differentiation

We are now prepared to deal with IMPLICIT functions. Our work so far has been with EXPLICIT functions.

Suppose we have a relationship such as $6xy^2+4xy+3y^2=0$. Clearly this is an implicit function (sec. 5–2). How do we calculate $\frac{dy}{dx}$? We differentiate straight through the equation with the formulas we have at hand.

A. Using formulas 5 and 7 (par. 5–11) we differentiate $6xy^2$ and $4xy$;

$$\text{formula (5): } \frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$$

Substituting $6x$ for u and y^2 for v in (5)

$$\frac{d(6xy^2)}{dx}=y^2\,\frac{d(6x)}{dx}+6x\,\frac{d(y^2)}{dx}$$

$$=6y^2+6x2y\frac{dy}{dx}$$

$$=6y^2+12xy\frac{dy}{dx}$$

Substituting $4x$ for u and y for v in (5)

$$\frac{d(4xy)}{dx}=y\frac{d(4x)}{dx}+4x\frac{dy}{dx}$$

$$=4y+4x\frac{dy}{dx}$$

B. Using formula 7 we differentiate $3y^2$;

$$\frac{d(3y^2)}{dx}=6y\frac{dy}{dx}$$

C. Putting terms together:

$$6y^2+12xy\frac{dy}{dx}+4y+4x\frac{dy}{dx}+6y\frac{dy}{dx}=0$$

D. Collect all terms containing $\frac{dy}{dx}$ on one side of the equation:

$$12xy\frac{dy}{dx}+4x\frac{dy}{dx}+6y\frac{dy}{dx}=-6y^2-4y$$

E. Solving for $\frac{dy}{dx}$;

$$\frac{dy}{dx}=\frac{-y(6y+4)}{12xy+6y+4x}=\frac{-y(3y+2)}{6xy+3y+2x}$$

Example 5–21.

Find $\frac{dy}{dx}$ if $3x^2y^3-xy+3=0$.

Without further explanation:

$$(6x)(y^3)+(3x^2)\left(3y^2\frac{dy}{dx}\right)-y-x\frac{dy}{dx}=0$$

$$9x^2y^2\frac{dy}{dx}-x\frac{dy}{dx}=y-6xy^3$$

$$\frac{dy}{dx}=\frac{y(1-6xy^2)}{x(9xy^2-1)}$$

5–13. Higher Order Differentiation

Once we have found the first derivative of a function, $\frac{dy}{dx}$ or y', we still have a function. If this new function is differentiable, we may find the second derivative of the original function. This process may be repeated any number of times, i.e., "n" times.

The successive derivatives may be noted in the following two ways for our purposes:

(1) $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$

(2) $y', y'', y''', \ldots, y^n$

The order of the derivative corresponds to the number of times the original function has been differentiated. The second derivative, $\frac{d^2y}{dx^2}$ or y'', is of the second order.

Example 5–22.

Find the first four derivatives of

$$y = x^5 + 3x^3 + 9x^2 + 7.$$

$$y' = 5x^4 + 9x^2 + 18x$$
$$y'' = 20x^3 + 18x + 18$$
$$y''' = 60x^2 + 18$$
$$y'''' = 120x$$

Example 5–23.

Find the first two derivatives of $x^2 - y^2 = 36$. Let us choose to do this example by implicit differentiation.

$$x^2 - y^2 = 36$$

(use theorem (3) paragraph 5–11)

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}, \text{ the first derivative.}$$

(use theorem (6) paragraph 5–11)

$$y'' = \frac{(y)(1) - (x)(y')}{y^2}$$

Substitute $\frac{x}{y}$ for y'

$$y'' = \frac{y - \frac{x^2}{y}}{y^2} = \frac{y^2 - x^2}{y^3}, \text{ the second derivative.}$$

Exercise 5–3.

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of the following functions.
   a. $y = \frac{x^2}{x+1}$
   b. $x^2 + y^2 = r^2$
   c. $y = \frac{x+1}{x-1}$
   d. $xy^2 - 1 = 0$

2. Perform the indicated operations.
   a. $y = (u^2+1)^3$ $u = (x-1)^2$
   (use theorems 7 and 8 paragraph 5–11)
   Find $\frac{dy}{dx}$: $\frac{dy}{dx} = 12(x-1)^{11} + 24(x-1)^7 + 12(x-1)^3$
   b. $y = \frac{1}{x}$ Find y''''. (Use theorem (2) paragraph 5–11.) $y'''' = \frac{24}{x^5}$
   c. $x = \frac{y^2-1}{2}$ Find $\frac{dy}{dx}$. (Transpose and solve for y, then use theorem 7 paragraph 5–11.) $\frac{dy}{dx} = \frac{1}{\pm\sqrt{2x+1}}$

5–14. Geometric Interpretation of the Derivative

We are familiar with terms like slope and tangent. We will use these terms here to illustrate the application of the derivative to geometry. (See fig. 5–4.)

We are aware from earlier work that the slope of a line is equal to the tangent of the angle that the line makes with a horizontal. For example, the slope of line P_1P_2 is equal to tan δ. Also calling the slope, m,

$$m_{P_1P_2} = \tan\delta = \frac{\Delta y}{\Delta x} \text{ where } \Delta y = y_2 - y_1,$$

and $\Delta x = x_2 - x_1$. We may also call this the average slope of $y = f(x)$ between the points P_1 and P_2.

Suppose now we allow the point P_2 to approach P_1 along $y = f(x)$ through P_3 and P_4. Notice that the slope of line P_1P_3, $m_{P_1P_3}$, $< m_{P_1P_2}$ and that $m_{P_1P_4} < m_{P_1P_3} < m_{P_1P_2}$. Notice also that these lines are approaching nearer and nearer to the line that is exactly tangent to $y = f(x)$ at P_1. We call this tangent line T. In fact in the limit as $P_2 \to P_1$, $\Delta x \to 0$ and we may write:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

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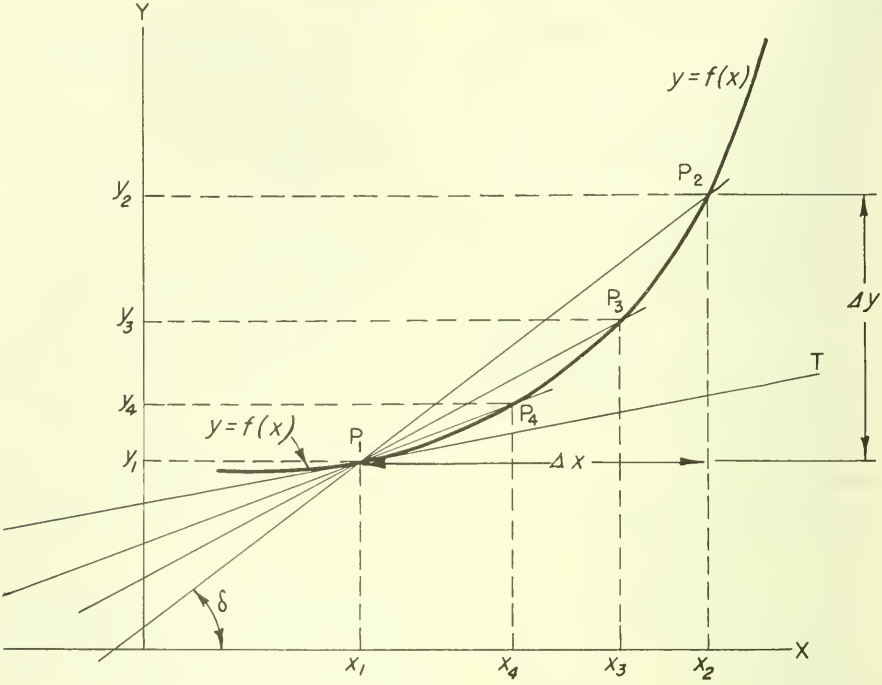


Figure 5–4.—Geometric interpretation of the derivative.

But this is exactly how we defined the derivative of a function earlier. More formally,

A. $\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$ and $\Delta x = x_2 - x_1$

B. $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$

$$= \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{dy}{dx}$$

We may say the derivative of a function evaluated at some point x_1, is equal to the slope of the function (and therefore the tangent) at point x_1.

A positive slope indicates the function is increasing as the independent variable increases and a negative slope indicates the function is decreasing as the independent variable increases.

Example 5–24.

Find the equations of the tangent and normal lines to the function $y = x^2 + 2x - 2$ at the point (1, 1), figure 5–5 A.

First we evaluate the slope at (1, 1).

$$\frac{dy}{dx} = m = 2x + 2 \text{ at } (1,\ 1),\ m = 4$$

remember the general form for the point slope form of a straight line to be;

$$\frac{y_1 - y_0}{x_1 - x_0} = m \text{ with } x_0 = 1 \text{ and } y_0 = 1$$

We have for the tangent line ($y_0 = 1$ and $x_0 = 1$); $\dfrac{y-1}{x-1} = 4$ or $y - 1 = 4x - 4$ and $y = 4x - 3$ which is the equation of the tangent line.

The slope of the normal line is the negative reciprocal of the slope of the tangent.

$$\therefore m = -\frac{1}{4}$$

$$\frac{y-1}{x-1} = -\frac{1}{4} \text{ or } 4y - 4 = -x + 1$$

and $4y = 5 - x$ which is the equation of the normal.

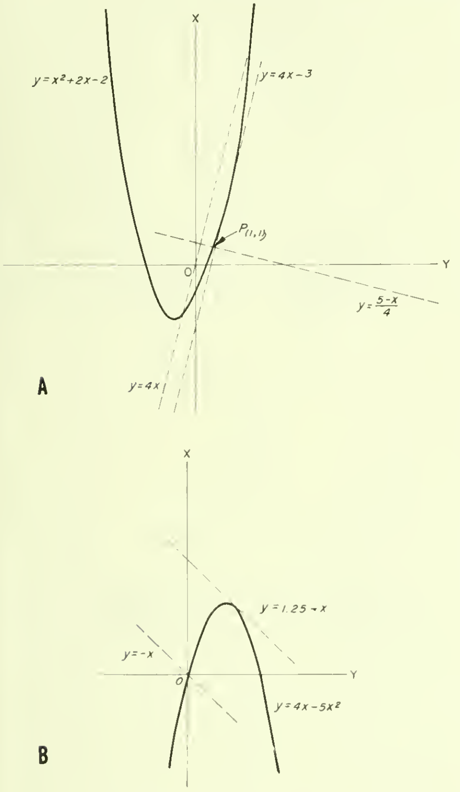


Figure 5–5.—(A) Graph of $y=x^2+2x-2$, $y=4x$, $y=4x-3$, and $y=\frac{5-x}{4}$ (B) Graph of $y=4x-5x^2$, $y=1.25-x$, and $y=-x$.

Example 5–25.

Find the point on the function $y=4x-5x^2$ where the tangent line has an inclination of $-45°$, figure 5–5 B.

The tangent of $-45°$ is -1 and is equal to the slope of the tangent line at the point desired.

$$\frac{dy}{dx}=4-10x=-1$$

$$10x=-5$$

$$x=\frac{1}{2}$$

When $x=\frac{1}{2}$, $y=\frac{3}{4}$ $\therefore$ the point is $\left(\frac{1}{2},\frac{3}{4}\right)$.

5–15. Critical Points; Maximum and Minimum

As the independent variable (x) passes through a range of values, the function (y) will in general pass through maximum and mimimum values. If the maximum value is the greatest value the function reaches over its defined range, we call it an ABSOLUTE MAXIMUM. If the minimum value is the least value the function reaches over its defined range, we call it an ABSOLUTE MINIMUM. If the maximum or minimum value we are investigating is not absolute, we call it a RELATIVE MAXIMUM or a RELATIVE MINIMUM. Any maximum or minimum value occurs at a CRITICAL POINT for the function.

In figure 5–6 we have plotted a function with four critical points in its range of definition. (We are saying that the function does not exist anywhere beyond those values that we have plotted.) The critical points, A and C, are maximum points. Point A is a RELATIVE MAXIMUM since in the vicinity of A, the function reaches its greatest value at A. Point C is an ABSOLUTE MAXIMUM for the function reaches its greatest value over its defined range at C. Similar reasoning permits us to call critical point B an ABSOLUTE MINIMUM, and critical point D a RELATIVE MINIMUM.

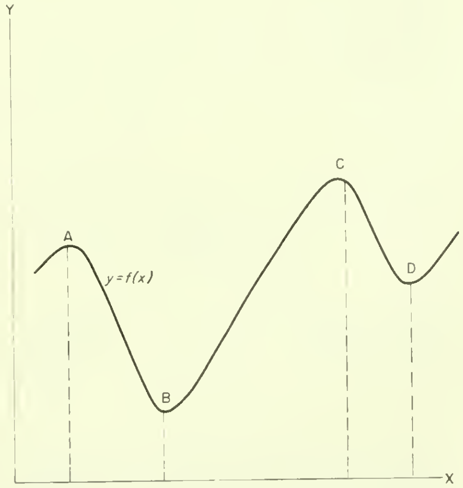


Figure 5–6.—A graph of $f(x)$ showing maxima and minima points.

We will examine the characteristics of a maximum point by looking at figure 5–7. As we proceed through increasing values of x, the slope (m) of the function changes from a positive value for $x < \text{A}$, to zero at $x = \text{A}$, and finally to a negative value for $x > \text{A}$. That is, $\frac{dy}{dx}$ changes from + to − as we pass through a maximum point. We must remember that $\frac{dy}{dx}$ is equal to the slope at any point on the function.

We may examine a minimum point in a similar manner by looking at figure 5–8. As we proceed through increasing values of x, the slope changes from a negative value for $x < \text{B}$, to zero at $x = \text{B}$, and finally the slope is positive for $x > \text{B}$. That is, $\frac{dy}{dx}$ changes from − to + as we pass through a minimum point.

In actual problems when we are investigating the sign of the derivative on either side of the critical point, we choose values for the independent variable very near to the value at the critical point. We must do this in order not to become involved with a critical point that may be close to the one in which we are interested.

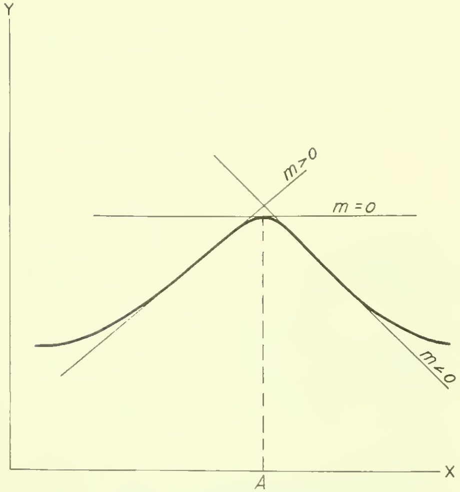


Figure 5–7.—The change of slope through a maximum point.

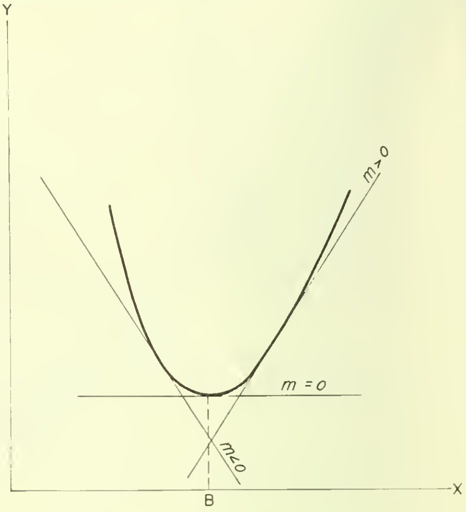


Figure 5–8.—The change of slope through a minimum point.

The second derivative also may be utilized to determine which type of critical point we have. Just as the first derivative $\frac{dy}{dx}$ of $y = f(x)$ provides us with the rate of change of y with respect to x, the second derivative, $\frac{d^2y}{dx^2}$, provides us with the rate of change of the first derivative. In other words, if the second derivative is positive, this indicates that the first derivative is increasing or becoming more positive. Conversely, if the second derivative is negative, it would indicate that the first derivative is decreasing or becoming less positive in nature. (More negative would mean the same as less POSITIVE.) Utilizing this information enables us to predict whether we have a maximum or a minimum point.

If at the point in question, we evaluate the second derivative of the function and find it to be positive we could conclude that our critical point was a minimum. Remember that a positive second derivative indicates an increasing first derivative and hence an increasing slope. At a minimum point the slope changes from − to + as we pass through the point which indicates an increase in value for the first derivative. Similarly, a negative second derivative indicates a decreasing first derivative and hence a decreasing slope. We would conclude that the critical point in question was a maximum point, for at a maximum point the slope changes from + to −, indicating a decrease in value for the first derivative.

Since all critical points occur where the slope is horizontal, it must be true that at any critical point, the first derivative is equal to zero.

So far we have dealt with only maximum and minimum points, but there is a third type of point called an INFLECTION POINT in which we will be interested (fig. 5–9). An inflection point occurs between maxi-

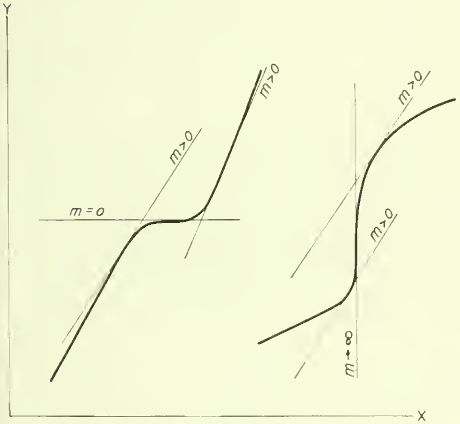


Figure 5–9. Change of slope through an inflection point.

mum and minimum points. That is, an inflection point occurs between critical points although an inflection point is not itself termed a critical point. In order to locate a point of inflection we set the SECOND DERIVATIVE equal to zero and solve the remaining equation. Once these values are determined we choose values slightly less than and slightly greater than this point and determine the signs of the second derivative on either side of this point. If the second derivative changes sign (+ to − or − to +), the point is a point of inflection.

Some functions have no critical points and no points of inflection. Such a function is $y = lnx$ (fig. 6–16).

Procedures for determining the locations and types of critical points and locations of inflection points are given below:

A. Calculate the first derivative and set the result equal to zero. From this equation determine which values of the independent variable will yield critical points.

B. The first derivative test: Choose a value for the independent variable slightly less than the point in question and note the sign of the first derivative. Choose a value slightly greater than the point in question and again note the sign of the first derivative.

Three possibilities may occur:

1. Signs change from + to −, indicating a maximum point.

2. Signs change from − to +, indicating a minimum point.

3. Signs remain the same, either − and − or + and +, implying a point of inflection (see D below).

C. The second derivative test: Calculate the second derivative and evaluate it at the critical point. Three possibilities result:

1. $y'' > 0$; minimum point
2. $y'' < 0$; maximum point
3. $y'' = 0$; test fails, use the first derivative test

D. If we are interested in locating points of inflection, we solve the equation resulting from $y'' = 0$. We examine the signs of y'' on either side of this point. If the signs of y'' are different, we have located a point of inflection.

The following examples will help to illustrate the general procedure.

Example 5–26.

Find and determine the nature of all critical points in the function $y = x^3 - 12x$, figure 5–10.

(1) Calculate the first derivative:

$$\frac{dy}{dx} = 3x^2 - 12$$

(2) Set the result equal to zero and solve for x.

$$3x^2 - 12 = 0$$
$$x^2 = 4 \text{ and } x = \pm 2$$

When $x = 2$, $y = 16$, and when $x = -2$, $y = 16$.

(3) First derivative test: Taking point $(2, -16)$, we evaluate the sign of the first derivative on either side of 2.

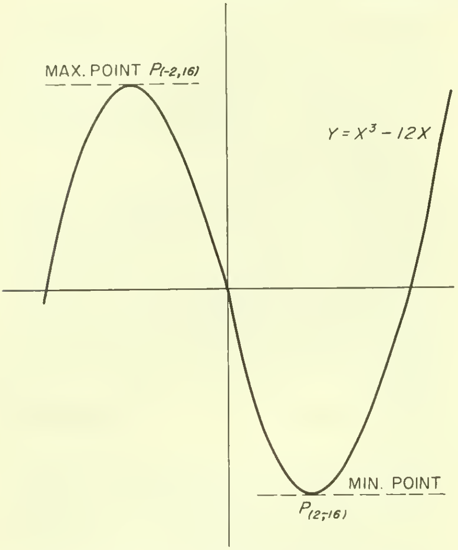


Figure 5-10.—Graph of $y=x^3-12x$ showing maxima and minima points.

$$x=\frac{3}{2};\ \frac{dy}{dx}=\frac{27}{4}-12 \quad \therefore \frac{dy}{dx}<0 \text{ (negative slope)}$$

$$x=\frac{5}{2};\ \frac{dy}{dx}=\frac{75}{4}-12 \therefore \frac{dy}{dx}>0 \quad \text{(positive slope)}$$

Since the sign of $\frac{dy}{dx}$ changes from $-$ to $+$, we conclude that the point $(2,-16)$ is a minimum point.

(4) Second derivative test using point $(2,-16)$:

$$\frac{d^2y}{dx^2}=6x \text{ so that for } x=2, \frac{d^2y}{dx^2}>0$$

$\therefore$ The second derivative tells us that the first derivative is increasing in value, i.e., going from $-$ to $+$ and point $(2,-16)$ is a mimimum point. We will repeat steps (3) and (4) for the point $(-2, 16)$ without explanation.

(3) $-\frac{5}{2}: \frac{dy}{dx}=\frac{75}{4}-12 \therefore \frac{dy}{dx}>0$ (positive slope)

$-\frac{3}{2}: \frac{dy}{dx}=\frac{27}{4}-12 \therefore \frac{dy}{dx}<0$ (negative slope)

$\frac{dy}{dx}$ changes from $+$ to $-$ $\therefore$ maximum point at $(-2, 16)$

(4) $$\frac{d^2y}{dx^2}=6x \text{ at } x=2\ \frac{d^2y}{dx^2}<0$$

$\therefore P_{-2,16}$ is a maximum point since $\frac{d^2y}{dx^2}$ is decreasing in value, i.e., going from $+$ to $-$.

Example 5-27.

Examine $y=\frac{t-1}{t^2}$ for all extreme (critical) points.

(1) $$\frac{dy}{dt}=\frac{t^2\frac{d(t-1)}{dt}-(t-1)\frac{d(t^2)}{dt}}{t^4}$$

$$\frac{dy}{dt}=\frac{t^2-2t^2+2t}{t^4}=\frac{2t-t^2}{t^4}=\frac{2-t}{t^3}$$

(2) $\frac{dy}{dt}=0$ so that $\frac{2-t}{t^3}=0$ and at $t=2$ we have an extreme point. When $t=2$, $y=\frac{1}{4}$.

(3) We will use the second derivative test to determine the nature of this point.

$$\frac{d^2y}{dt^2}=\frac{t^3\frac{d(2-t)}{dt}-(2-t)\frac{d(t^3)}{dt}}{t^6}$$

$$\frac{d^2y}{dt^2}=\frac{-t^3-6t^2+3t^3}{t^6}=\frac{2t-6}{t^4}$$

When $t=2$, $\frac{d^2y}{dt^2}=\frac{4-6}{16}=-\frac{1}{8}$

$\therefore$ The point $t=2$ and $y=\frac{1}{4}$ is a maximum.

(4) We now wish to locate any points of inflection. We set the second derivative equal to zero;

$$\frac{d^2y}{dt^2}=\frac{2t-6}{t^4}=0 \qquad \therefore t=3 \text{ and } y=\frac{2}{9}$$

We choose values of t slightly less than and slightly greater than three and note the signs of $\frac{d^2y}{dt^2}$.

$$t=2.5: \frac{2(2.5)-6}{(2.5)^4} \qquad \frac{d^2y}{dt^2}<0 \text{ (negative)}$$

$$t=3.5: \frac{2(3.5)-6}{(3.5)^4} \qquad \frac{d^2y}{dt^2}>0 \text{ (positive)}$$

As the second derivative changes sign from $-$ to $+$, $t=3$ and $y=\frac{2}{9}$ is an inflection point.

Example 5–28.

Determine all critical points of the function $y=(z-1)^3$.

(1) $\frac{dy}{dz}=3(z-1)^2$

(2) $\frac{dy}{dz}=0=3(z-1)^2$

$\therefore z=1$ and $y=0$ is the point to be examined.

(3) We first attempt the second derivative test:

$\frac{d^2y}{dz^2}=6(z-1)$, but when $z=1$ the second derivative is zero. Thus, the test fails and we must go back to the first derivative test.

(4) We will test the signs of the first derivative at $z=\frac{3}{2}$ and $z=\frac{1}{2}$ which are on either side of $z=1$.

$$z=\frac{1}{2}: \frac{dy}{dz}>0$$
$$z=\frac{3}{2}: \frac{dy}{dz}>0$$

Therefore, the first derivative is zero when $z=1$, but the SIGN of the derivative on either side of $z=1$ is positive. Thus, the point $z=1$ and $y=0$ is neither a maximum nor a minimum point.

If we set the second derivative equal to zero, we get $z=1$. Checking the sign of the second derivative on either side of $z=1$, we obtain;

$$z=\frac{1}{2}: \frac{d^2y}{dz^2}<0$$
$$z=\frac{3}{2}: \frac{d^2y}{dz^2}>0$$

$\therefore$ The point $z=1$ and $y=0$ is an inflection point.

Example 5–29.

In this series circuit (fig. 5–11) we have the constant source E, the constant source resistance r, the current i, and the load resistance R. We wish to calculate what load resistance is necessary to permit maximum power transmission from the source to the load.

Power used in a resistor such as load resistor R is given by the formula $P=i^2R$ where P is the power in watts, i is the current in amperes, and R is the load resistance in ohms. In this equation P is the dependent variable and is a function of the TWO quantities, i and R. Since we wish to determine the maximum power utilized by the load

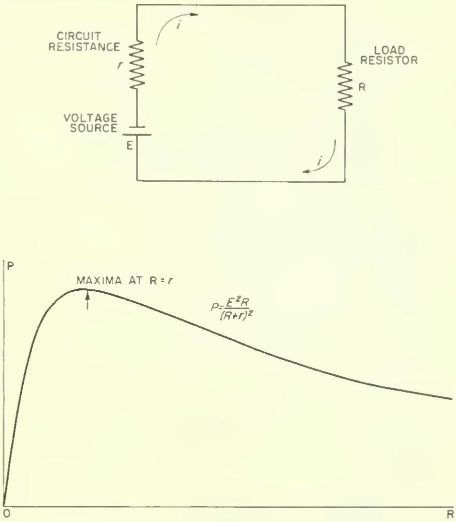


Figure 5–11.—Series circuit for example 5–29.

resistance, we must look for some relationship between i and R in order to get P as a function of R alone.

The total current in the circuit can be found from the relationship $i=\frac{E}{r+R}$. We have already stated that both E and r are constants so we have $i=f(R)$. We may now substitute $\frac{E}{r+R}$ for i in the power expression, i^2R.

This gives $P=\left(\frac{E}{r+R}\right)^2R=\frac{E^2R}{(r+R)^2}$. We wish to maximize the power with respect to the load resistance, so we take $\frac{dP}{dR}$ and set it equal to zero. Remembering that E and r are constant;

$$\frac{dP}{dR}=\frac{d\left(\frac{E^2R}{(r+R)^2}\right)}{dR}$$

Using theorem (6) paragraph 5–11 let $u=(E^2R)$. and $v=(r+R)^2$.

Substituting in

$$\frac{d\left(\frac{u}{v}\right)}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$\frac{dP}{dR}=\frac{(r+R)^2\frac{d(E^2R)}{dR}-E^2R\frac{d(r+R)^2}{dR}}{[(r+R)^2]^2}$$

Using theorem (5) paragraph 5–11 let $u=E^2$ and $v=R$.
Substitute in

$$\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{dv}{dx};\ \frac{d(E^2R)}{dR}=R\ 0+E^2\ 1=E^2$$

NOTE: E is a constant and $\frac{dR}{dR}=1$

Using theorem (2) paragraph 5–11 let $x=(r+R)$ and $n=2$
Substitute in

$$\frac{dx^m}{dx}=nx^{m-1};\ \frac{d(r+R)^2}{dR}=2(r+R)^{2-1}$$

$$\frac{dP}{dR}=\frac{(r+R)^2E^2-E^2R(2)\ (r+R)}{(r+R)^4}$$

Cancel one $(r+R)$ throughout;

$$\frac{dP}{dR}=\frac{(r+R)E^2-2E^2R}{(r+R)^3}$$

Set $\frac{dP}{dR}=0$;

$$(r+R)E^2-2E^2R=0$$
$$r+R=2R$$
$$r=R$$

We see the critical point for the load resistance occurring when the source resistance is equal to the load resistance.

We will attempt to confirm that this will yield a maximum power utilization with the second derivative test. If the second derivative is negative for this value of R, $(R=r)$, we will have a maximum. You may say in a practical way that it is not a minimum for it should be evident minimum power utilization will occur when the circuit is opened. This would still leave the possibility that our critical point may be an inflection point!

Proceeding:

$$\frac{dP}{dR}=\frac{(r+R)E^2-2E^2R}{(r+R)^3}$$

We will use formulas 2, 4, and 6 for differentiating the expression for $\frac{dP}{dR}$.

First simplify;

$$\frac{dP}{dR}=\frac{rE^2+E^2R-2E^2R}{(r+R)^3}=\frac{rE^2-E^2R}{(r+R)^3}$$

and remember that r and E are constants;

$$\frac{d^2P}{dR^2}=\frac{d}{dR}\left(\frac{rE^2-E^2R}{(R+r)^3}\right)$$

Treat the above expression as $y=\frac{u}{v}$ and take

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2};$$

let $u=(rE^2-E^2R)$ and $v=(R+r)^3$

$$\frac{d^2P}{dR^2}=\frac{(R+r)^3\frac{d}{dR}(rE^2-E^2R)-(rE^2-E^2R)\frac{d}{dR}(R+r)^3}{[(R+r)^3]^2}$$

$$\frac{d}{dR}(rE^2-E^2R)=\frac{d}{dR}(rE^2)+\frac{d}{dR}(-E^2R)=-E^2\frac{dR}{dR}=-E^2$$

Treat the first term of the above expression as $y=c$ and take $\frac{dy}{dx}=0$; let $c=(rE^2)$;

treat the second term of the above expression as $y=cv$ and take $\frac{dy}{dx}=c\frac{dv}{dx}$; let $c=-E^2$ and $v=R$.

$$\frac{d}{dR}(R+r)^3=3(R+r)^2$$

Treat the above expression as $y=x^n$ and take $\frac{dy}{dx}=nx^{n-1}$; let $n=3$ and $x=(R+r)$

Combining the above terms,

$$\frac{d^2P}{dR^2}=\frac{(R+r)^3(-E^2)-(rE^2-E^2R)\,(3)\,(R+r)^2}{(R+r)^6}$$

Cancel $(r+R)^2$;

$$\frac{d^2P}{dR^2}=\frac{(r+R)\,(-E^2)-3\,(rE^2-E^2R)}{(r+R)^4}$$

$$\frac{d^2P}{dR^2}=\frac{-E^2r-E^2R-3E^2r+3E^2R}{(r+R)^4}$$

Simplify:

$$\frac{d^2P}{dR^2}=\frac{2E^2R-4E^2r}{(r+R)^4}$$

Now substitute the value for R at the critical point in question ($R=r$):

$$\frac{d^2P}{dR^2}=\frac{2E^2r-4E^2r}{(2r)^4}=-\frac{2E^2r}{16r^4}$$

$$\frac{d^2P}{dR^2}=-\frac{E^2}{8r^3}\therefore\frac{d^2P}{dR^2}<0$$

and because the second derivative is less than zero (that is, negative) we have verified that when the load resistance R is equal to the source resistance r, power utilization will be a maximum.

A plot of the power expression $P=\dfrac{E^2R}{(r+R)^2}$, showing the maximum point at $R=r$ is given in figure 5–11.

Example 5–30.

A series resistance, capacitance, and inductance circuit with an a-c input is shown in figure 5–12A. Assume we have a constant input voltage, a constant inductive reactance (X_L) and a constance resistance, R. We wish to determine what value of capacitive reactance (X_c) will give the least opposition to current flow in this circuit. Opposition to current flow in an a-c circuit is called impedance and is designated by the symbol Z.

The expression for the impedance in this circuit is given by;

$$Z=\sqrt{R^2+(X_L-X_c)^2} \qquad \text{(See figure 5–12, B)}$$

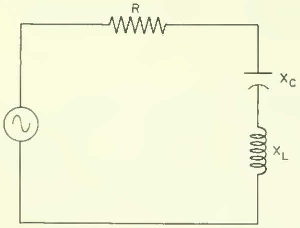
We want the minimum value of Z with respect to X_c. Remember that R and X_L are to be held constant.

We may see by inspection that the minimum value for Z occurs when $X_c=X_L$. We will proceed to verify this by the use of the calculus.

The procedure will be the same as in the preceding problems;

(1) Calculate the first derivative and set the result equal to zero to determine what values of the independent variable (X_c) will yield a critical point.

(2) Use either the first or second derivative test to determine whether the point is a maximum or minimum.



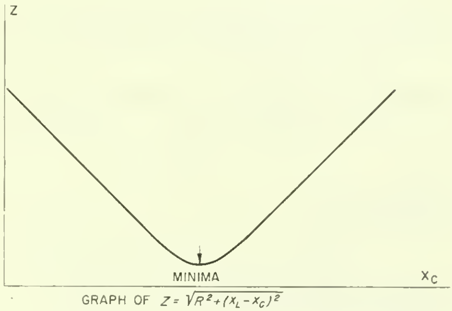


Figure 5–12.—(A) Series *RLC* circuit. (B) Graph of $Z=\sqrt{R^2+(X_L-X_C)^2}$.

Proceeding;

$$Z=[R^2+(X_L-X_c)^2]^{1/2}$$

(We treat this expression as $y=u^n$ and calculate $\dfrac{dy}{dx}=nu^{n-1}\dfrac{du}{dx}$, formula (7) paragraph 5–11.)

$$\frac{dZ}{dX_c}=\frac{1}{2}[R^2+(X_L-X_c)^2]^{-1/2}\,2(X_L-X_c)(-1)$$

Collect terms and equate to zero:

$$\frac{X_c-X_L}{[R^2+(X_L-X_c)^2]^{1/2}}=0$$

$$\therefore X_c=X_L$$

The critical point occurs when X_c is equal to X_L. Now we must determine whether this yields a maximum or a minimum value for Z.

We will use the second derivative test.

$$\frac{dZ}{dX_c}=\frac{X_c-X_L}{[R^2+(X_L-X_c)^2]^{1/2}}$$

Treat this expression as $y=\frac{u}{v}$ and take

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2};\ \text{let } u=(X_c-X_L) \text{ and } v=[R^2+(X_L-X_c)^2]^{\frac{1}{2}}$$

$$\frac{d^2Z}{dX_c^2}=\frac{[R^2+(X_L-X_c)^2]^{\frac{1}{2}}\frac{d}{dX_c}(X_c-X_L)-(X_c-X_L)\frac{d}{dX_c}[R^2+(X_L-X_c)^2]^{\frac{1}{2}}}{\{[R^2+(X_L-X_c)^2]^{\frac{1}{2}}\}^2}$$

$$=\frac{[R^2+(X_L-X_c)^2]^{\frac{1}{2}}(1)-(X_c-X_L)\frac{d}{dX_c}[R^2+(X_L-X_c)^2]^{\frac{1}{2}}}{\{[R^2+(X_L-X_c)^2]^{\frac{1}{2}}\}^2}$$

Treat the expression $\frac{d}{dX_c}[(R^2+(X_L-X_c)^2]^{\frac{1}{2}}$ as $y=u^n$

and take $\frac{dy}{dx}=nu^{n-1}\frac{du}{dx}$; let $u=R^2+(X_L-X_c)^2$ and $n=\frac{1}{2}$

$$\frac{d}{dX_c}[R^2+(X_L-X_c)^2]^{\frac{1}{2}}$$

$$=\frac{1}{2}[R^2+(X_L-X_c)^2]^{-\frac{1}{2}}\frac{d}{dX_c}[R^2+(X_L-X_c)^2]$$

Treat the expression $\frac{d}{dX_c}[R^2+(X_L-X_c)^2]$ as

$$\frac{dR^2}{dX_c}+\frac{d}{dX_c}(X_L-X_c)^2$$

R and X_L are constants.

$$\frac{d}{dX_c}[R^2+(X_L-X_c)^2]=0+\frac{d}{dX_c}(X_L-X_c)^2$$

Treat the expression $\frac{d}{dX_c}(X_L-X_c)^2$ as $y=u^n$ and take

$$\frac{dy}{dx}=nu^{n-1}\frac{du}{dx};\ \text{let } u=X_L-X_c \text{ and } n=2$$

$$\frac{d}{dX_c}(X_L-X_c)^2=2(X_L-X_c)^{2-1}\frac{d(-X_c)}{X_c}=2(X_L-X_c)(-1)$$

Substituting (3) in (2)

$$\frac{d}{dX_c}[R^2+(X_L-X_c)^2]^{\frac{1}{2}}=\frac{1}{2}[R^2+(X_L-X_c)^2]^{-\frac{1}{2}}2(X_L-X_c)(-1)$$

$$=\frac{-\frac{1}{2}(2)(X_L-X_c)}{[R^2+(X_L-X_c)^2]^{\frac{1}{2}}}$$

Substituting (4) in (1)

$$\frac{d^2Z}{dX_c^2}=\frac{[R^2+(X_L-X_c)^2]^{\frac{1}{2}}(1)-(X_c-X_L)\frac{-\frac{1}{2}(2)(X_L-X_c)}{[R^2+(X_L-X_c)^2]^{\frac{1}{2}}}}{\{[R^2+(X_L-X_c)^2]^{\frac{1}{2}}\}^2}$$

$$=\frac{[R^2+(X_L-X_c)^2]^{\frac{1}{2}}-\frac{(X_L-X_c)(X_L-X_c)}{[R^2+(X_L-X_c)^2]^{\frac{1}{2}}}}{[R^2+(X_L-X_c)^2]^1}$$

$$\frac{d^2Z}{dX_c{}^2}=\frac{R^2+(X_L-X_c)^2-(X_L-X_c)^2}{[R^2+(X_L-X_c)^2]^{\frac{3}{2}}}$$

We substitute our critical point value of X_c, $X_c=X_L$:

$$\frac{d^2Z}{dX_c{}^2}=\frac{R^2+(X_L-X_L)^2-(X_L-X_L)^2}{[R^2+(X_L-X_L)^2]^{\frac{3}{2}}}=\frac{R^2}{R^3}$$

$$\frac{d^2Z}{dX_c{}^2}=\frac{1}{R}\ or\ \frac{d^2Z}{dX_c{}^2}>0$$

Since $\frac{d^2Z}{dX_c{}^2}>0$, we have a MINIMUM value for the impedance (Z) when $X_c=X_L$.

Looking at the expression for Z,

$$Z=\sqrt{R^2+(X_L-X_c)^2},$$

the minimum value for Z is R. When this occurs, we say the circuit is in a resonant condition.

In an a-c series RLC circuit we have found that resonance occurs when $X_c=X_L$.

5–16. Rectilinear Motion

Rectilinear motion is simply motion along a straight line. Motion of this type may have either of two possible directions at any instant. One of these directions is arbitrarily chosen as positive and thereby fixes the other direction as negative.

In figure 5–13 let us assume we begin our movements from the origin marked O. If motion is from left to right, O to A, B to A, A′ to B′, etc., it will be considered +, and motion from right to left, O to A′, B′ to A′, B to O, etc., will be considered negative.

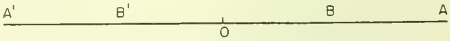


Figure 5–13.—Rectilinear motion.

Suppose we move a distance OB in some time interval Δt. Our AVERAGE rate of change of distance with respect to time would be; $\frac{OB}{\Delta t}$. We term this the average velocity during the time Δt. If we had called the distance OB, Δs, instead, we would see that the average velocity $\bar{v}$ (called "v bar") is equal to $\frac{\Delta s}{\Delta t}$.

Allowing $\Delta t \rightarrow 0$, we may define the instantaneous rate of change of distance with respect to time as $\frac{ds}{dt}$ for by the definition of the derivative we know that $\lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$. We term this limit the instantaneous velocity, i.e., $v = \frac{ds}{dt}$.

Note that the instantaneous velocity, v, is written without the bar above the v.

Proceeding in a like manner, we may define the average rate of change of the velocity with respect to time as the average acceleration, i.e., $\bar{a} = \frac{\Delta v}{\Delta t}$.

The instantaneous acceleration expression may be arrived at by again allowing $\Delta t \rightarrow 0$. We have $\lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$ and is equal to the instantaneous acceleration.

So far we have; (1) $v = \frac{ds}{dt}$ and (2) $a = \frac{dv}{dt}$, but notice that $\frac{dv}{dt} = \frac{d^2s}{dt^2}$. We may elaborate and say that the velocity is equal to the first derivative of the distance with respect to time, and the acceleration is equal to the second derivative of the distance with respect to time.

Velocity in rectilinear motion is: $v = \frac{ds}{dt}$.

Acceleration in rectilinear motion is;

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Example 5–31.

Given that a particle moves in a straight line according to the equation $s = 3t + 8t^2$, where s is in feet and t in seconds. (Fig. 5–14.)

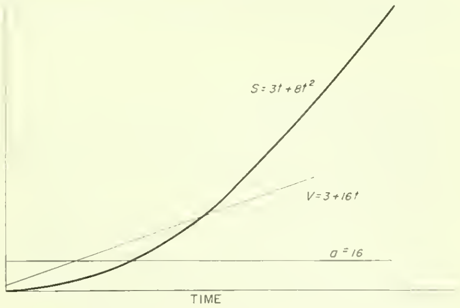


Figure 5–14.—Graph of $s=3t+8t^2$, $a=16$, and $v=3+16t$.

(a) What distance is traveled during the first three seconds? s is unknown and $t=3$.

$$\therefore \quad s = 3(3) + 8(3)^2 = 9 + 72 = 81 \text{ ft.}$$

(b) What is the AVERAGE velocity during the fourth second?

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{s_4 - s_3}{t_4 - t_3} = \frac{3(4) + 8(4)^2 - [3(3) + 8(3)^2]}{4 - 3}$$

$$\bar{v} = \frac{140 - 81}{1} = 59 \text{ ft/sec}$$

(c) What are the instantaneous velocities when $t=3$ secs. and $t=4$ secs.?

$$v = \frac{ds}{dt} = \frac{d(3t + 8t^2)}{dt} = 3 + 16t$$

$$v_3 = 3 + 16(3) = 51 \text{ ft/sec}$$
$$v_4 = 3 + 16(4) = 67 \text{ ft/sec}$$

(d) Calculate the average acceleration during the third second.

$$\bar{a} = \frac{v_3 - v_2}{t_3 - t_2} = \frac{3 + 16(3) - [3 + 16(2)]}{3 - 2} = 16 \text{ ft/sec}^2$$

(e) Calculate the instantaneous acceleration at $t=2$ seconds.

$$a = \frac{dv}{dt}\frac{d^2s}{dt^2} = \frac{d(3 + 16t)}{dt} = 16 \text{ ft/sec}^2$$

It is important to note that in this example the instantaneous and average accelerations are equal.

This is always true when the acceleration is a constant value.

5–17. Curvilinear Motion

Curvilinear motion is motion along a curve in a plane. Figure 5–15 illustrates curvilinear motion from P through Q to R.

Curvilinear motion may most easily be thought of as two simultaneously occurring rectilinear motions. One portion is the rectilinear motion of the particle along the y axis and the other is the rectilinear motion of the particle along the x axis. The vector addition of these two motions yields curvilinear motion as in figure 5–15.

It would be well to mention at this point that both velocity and acceleration are vector quantities, i.e., both have magnitude and direction. As such the resultant velocity and acceleration vectors may be resolved into components and the components may be combined by vector addition into resultants. If these ideas are not immediately apparent, review of *Math, Volume 2*, NavPers 10071–A, section on vectors, is recommended. We will identify vectors in this text by bold face print with a small arrow above the letter. $\vec{A}$ would mean "the vector A".

In general, curvilinear motion will be described by some relations such as $\vec{y}=f(t)$ and $\vec{X}=g(t)$. By these we mean the $\vec{y}$ and $\vec{x}$ are both functions of the variable t, but they are different functions of t. The velocity vector in the y direction is given by:

$\vec{v}_y=\frac{d\vec{y}}{dt}$, and in the x direction $\vec{v}_x=\frac{d\vec{x}}{dt}$.

The resultant velocity is the vector addition of the two component velocities.

$$\vec{v}=\vec{v}_x+\vec{v}_y=\frac{d\vec{x}}{dt}+\frac{d\vec{y}}{dt}$$

The direction of the resultant velocity can be found from $\tan\theta=\frac{v_y}{v_x}$. In this relationship we may drop the vector notation and use magnitudes only. Note that $\tan\theta=\frac{v_y}{v_x}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{dy}{dx}$ and we see that the velocity vector at any point lies along the tangent to the curve at that point and points in the direction of travel along the curve (fig. 5–16).

The speed of a particle traveling along a curve is the magnitude of the velocity vector.

$$\text{Speed}=|\vec{v}|=\sqrt{\vec{v}_x{}^2+\vec{v}_y{}^2}$$

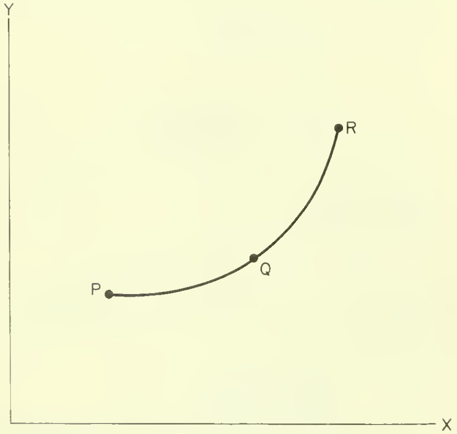


Figure 5–15.—Curvilinear motion.

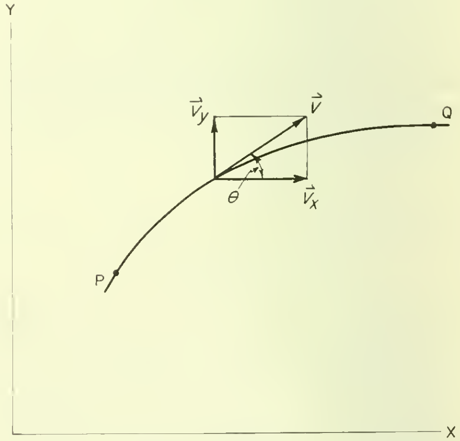


Figure 5–16.—Velocity in curvilinear motion.

The acceleration vector in the y direction is given by $\vec{a}_y=\frac{d\vec{v}_y}{dt}$ and in the x direction $\vec{a}_x=\frac{d\vec{v}_x}{dt}\cdot$ We may then calculate the resultant acceleration;

$$\vec{a}=\vec{a}_x+\vec{a}_y=\frac{d\vec{v}}{dt}=\frac{d\vec{v}_x}{dt}+\frac{d\vec{v}_y}{dt}=\frac{d^2\vec{x}}{dt^2}+\frac{d^2\vec{y}}{dt^2}$$

The direction of the acceleration vector is obtained by $\tan\phi=\frac{a_y}{a_x}\cdot$ Here again we use the magnitudes only (fig. 5–17).

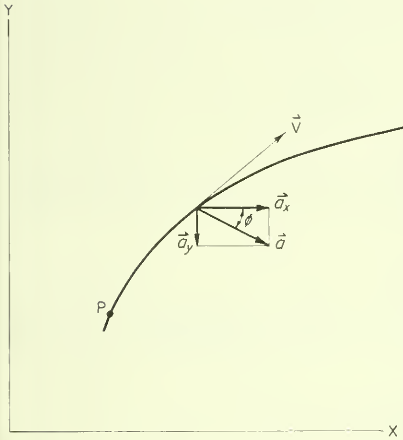


Figure 5–17.—Acceleration in curvilinear motion.

In general, the resultant acceleration vector will not lie along the same line as the resultant velocity vector. It is the acceleration that causes the motion to change directions and hence to be curved so that should the velocity and acceleration vectors lie along the same line, the motion would be along a straight line.

The magnitude of the acceleration may be obtained from

$$a=|\vec{a}|=\sqrt{\vec{a}_x{}^2+\vec{a}_y{}^2}$$

Example 5–32.

Given that $\vec{y}=f(t)$ and $\vec{x}=g(t)$. If $f(t)=t^2-1$ and $g(t)=t^3+t^2-3$, calculate $\vec{x}$, $\vec{y}$, $\vec{v}_x$, $\vec{v}_y$, $\vec{v}$, v, $\vec{a}_x$, $\vec{a}_y$, $\vec{a}$, and a at $t=2$. $\vec{x}$ and $\vec{y}$ are in feet and t is in seconds. Note here that when $t=0$, $\vec{y}=-1$ and $\vec{x}=-3$.

(a) $x=t^3+t^2-3$ and $y=t^2-1$

$\therefore$ when $t=2$; $x=9$ ft. and $y=3$ ft.

Select an arbitrary origin at (0, 0) so that at $t=2$ we are 9 feet to the right and 3 feet above our origin.

(b) $\vec{v}_x=\frac{d\vec{x}}{dt}=\frac{d}{dt}(t^3+t^2-3)=3t^2+2t$

$\therefore$ when $t=2$; $\vec{v}_x=16$ ft/sec.

The direction of $\vec{v}_x$ is to the right. $\rightarrow$

(c) $\vec{v}_y=\frac{d\vec{y}}{dt}=\frac{d}{dt}(t^2-1)=2t$

$\therefore$ when $t=2$; $\vec{v}_y=4$ ft/sec

The direction of $\vec{v}_y$ is vertically up. $\uparrow$

(d) $v=|\vec{v}|=\sqrt{\vec{v}_x{}^2+\vec{v}_y{}^2}=\sqrt{256+16}=\sqrt{272}$

$v=16.5$ ft/sec

(e) We now have the magnitude of $\vec{v}$ from part (d). We need only the direction to complete the calculation for $\vec{v}$.

$$\tan\theta=\frac{|\vec{v}_y|}{|\vec{v}_x|}=\frac{v_y}{v_x}=\frac{4}{16}=\frac{1}{4}$$

$$\tan\theta=.25 \qquad \theta=14.05°$$

and $\vec{v}=16.5$ ft/sec $\underline{\nearrow\!\!\!\swarrow\ 14.05°}$

In order to measure this angle we begin at the x axis and measure the degrees counterclockwise.

(f) $\vec{a}_x=\frac{d\vec{v}_x}{dt}=\frac{d}{dt}(3t^2+2t)=\frac{d^2\vec{x}}{dt^2}=6t+2$

$\therefore \vec{a}_x=14$ ft/sec^2 $\rightarrow$

(g) $\vec{a}_y=\frac{d\vec{v}_y}{dt}=\frac{d}{dt}(2t)=\frac{d^2\vec{y}}{dt^2}=2$

$\therefore \vec{a}_y=2$ ft/sec^2

(h) $a=|\vec{a}|=\sqrt{\vec{a}_x{}^2+\vec{a}_y{}^2}=\sqrt{196+4}=\sqrt{200}$

$\therefore a=14.15$ ft/sec^2

(i) We have the magnitude of $\vec{a}$ from part (h) and we need only the direction to complete the calculation.

$$\tan\phi=\frac{|\vec{a}_y|}{|\vec{a}_x|}=\frac{a_y}{a_x}=\frac{2}{14}=\frac{1}{7}=.143$$

so that $\phi=8.14°$

$\vec{a}=14.15$ ft/sec^2 $\underline{\nearrow\!\!\!\swarrow\ 8.14°}$

Exercise 5–4.

A. Determine the location and nature of the critical points in the following functions:

1. $y=\frac{x^2+1}{x}$

2. $y=\frac{x}{(x-1)^{1/2}}$

3. A varying current is passed through a resistor whose resistance changes according to $r=\frac{1}{i^2+1}$. For what current will the voltage drop across the resistor reach its maximum value and what is this maximum value? The relationship between voltage, current, and resistance is $V=iR$ (fig. 5–18).

B. Problems involving rectilinear and curvilinear motion.

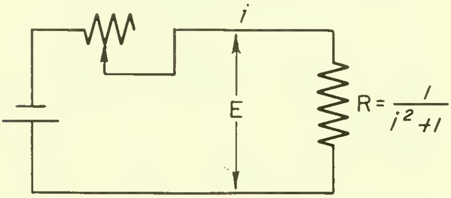


Figure 5–18.—Find E_{max} and i for E_{max}.

1. For any body falling under the influence of gravity, the equation $\vec{y}=\vec{v}_0t+\frac{1}{2}\vec{a}t^2$ describes its motion. $\vec{v}_0$ is the object's initial velocity, $\vec{a}$ is the acceleration due to gravity, and $\vec{y}$ is the distance the object has fallen with respect to any chosen reference. With $\vec{y}$ given in feet and t in seconds, (a) when will the falling body halve the distance to the earth if it began its journey at 64 feet? (b) What is its velocity at this time? Time $t=0$ seconds is when the body is at 64 feet. Hint: choose your reference at 64 feet and pick all velocities and accelerations down as positive.

2. A particle moves along the x axis according to $x=3t^2-t^3$. Take positive values of x and its derivatives as motion to the right. (a) When will the particle come to rest once it has begun its motion? (b) In what direction is the acceleration vector pointing at this time? (c) In which direction is the particle moving at $t=3$ sec?

3. Given the relations $\vec{y}=t^2$ and $\vec{x}=2t$ describing curvilinear motion. At $t=3$ seconds calculate the particles position, $\vec{v}_x$, $\vec{v}_y$, and $\vec{v}$ and $\vec{a}_x$, $\vec{a}_y$, and $\vec{a}$. x and y are in feet.

5–18. Related Rate Problems

Many physical problems exist in which all the variables are functions of time. In addition the problems are of such a type that relationships between the various variables are also available. Since these problems so frequently occur, they are called RELATED RATE PROBLEMS.

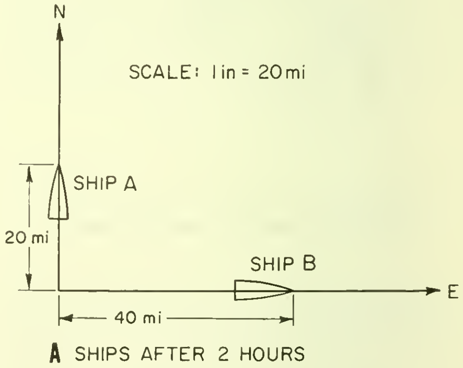
The following several examples will illustrate the techniques used to solve this group of problems.

Example 5–33.

Two ships begin from the same point at the same time. Ship A travels due north at 10 knots and ship B travels due east at 20 knots. How fast are the ships separating at two hours after the trips began? (See figures 5–19A and 5–19B.)

We are interested in $\frac{dz}{dt}$ and will proceed with this in mind.

Let us check the two requirements for a related rate problem to see if they are fulfilled in this case.



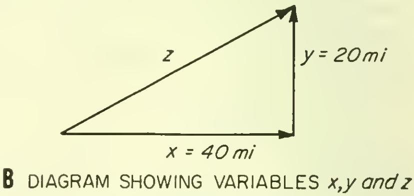


Figure 5–19.—Example 5–33.

(1) Are all variables functions of time? Yes, since x, y, and z vary with time.

(2) Does there exist some relationship between all variables? Yes, as $z^2=x^2+y^2$.

(A) Write the relationship between the variables.

$$z^2=x^2+y^2$$

(B) Differentiate this expression with respect to time.

$$2z\frac{dz}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}$$

(C) Substitute all known quantities.

$$\frac{dx}{dt}=20 \text{ knots; } \frac{dy}{dt}=10 \text{ knots; } t=2 \text{ hours}$$

$$x=40;\ y=20$$

We get z from $z=\sqrt{x^2+y^2}=\sqrt{2000}$ and $z=20\sqrt{5}$

$\therefore\ 2(20\sqrt{5})\frac{dz}{dt}=2(40)(20)+2(20)(10)$ and $\frac{dz}{dt}=\frac{50}{\sqrt{5}}$

$=10\sqrt{5}$ knots.

Example 5–34.

If a sphere expands at the rate of 15 ft³/min, what will be the rate of change of the radius when the diameter reaches 10 feet?

(A) Volume of a sphere $=\frac{4}{3}\pi R^3$

(B) $\frac{dv}{dt}=\frac{4}{3}\pi 3R^2\frac{dR}{dt}$

(C) $15=\frac{4}{3}\pi(3)(25)\frac{dR}{dt}$

$$\therefore\quad \frac{dR}{dt}=\frac{3}{20\pi}\text{ ft/min}$$

Example 5–35.

A right circular cone (fig. 5–20) has a height of 12 inches and a base diameter of 8 inches. If the cone is inverted and we pour water into the base of the cone causing the radius at the water level to change at a rate of $\frac{1}{3}$ in/min, at what rate will the volume of water be changing when the radius is 2 inches?

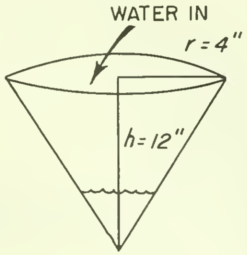


Figure 5–20.—Filling a right circular cone.

(A) The volume of a right circular cone is given by $v=\frac{1}{3}\pi r^2h$. We are interested in the rate of change of volume and are given the rate of change of the radius. We must find an expression for h in terms of r.

To find this expression we look at the geometry of the cone. The sides have a constant slope $=\frac{h}{r}=\frac{12}{4}=3$.

$h=3r$ at any position along the altitude.

(B) $\therefore v=\frac{1}{3}\pi r^2(3r)=\pi r^3$

$$\frac{dv}{dt}=3\pi r^2\frac{dr}{dt} \text{ and } r=2 \text{ in,}$$

$$\frac{dr}{dt}=\frac{1}{3}\text{ in/min.}$$

(C) $\frac{dv}{dt}=3\pi(2)^2\frac{1}{3}=4\pi$ in³/min.

Exercise 5–5.

1. A vertically mounted 12-inch cylinder is $\frac{2}{3}$ filled with water. The level of the water is dropping at the rate of $\frac{1}{2}$ in/min. If the cylinder has a radius of 4 inches, at what rate is water escaping from the cylinder?

2. A ladder 20 feet long rests with its top most portion against a building and 16 feet from the ground. The ladder slips and begins to slide down the building at a rate of 2 ft/sec. A bucket of paint is resting 4 feet from the base of the ladder on the ground when the ladder begins to slip. At what rate

is the base of the ladder moving when the bucket of paint is hit by the base of the ladder?

3. A curve is described by $3x = y^2 - 2y + 1$. What is the value of the abscissa, x, when the ordinate is changing at the rate of 3 units/sec and the abscissa is changing at 2 units/sec?

5-19. Circuit Devices

Applications of the derivative can be found in any field of science. We will investigate several applications here as associated with electricity and circuitry.

As we've found in our basic electricity courses, any current-carrying conductor has a magnetic field associated with it (fig. 5-21). If for any reason these magnetic lines of force are altered or changed in number, we have produced an induced emf (electromotive force) in the conductor with such polarity as to attempt to restore this magnetic field to its original undisturbed condition.

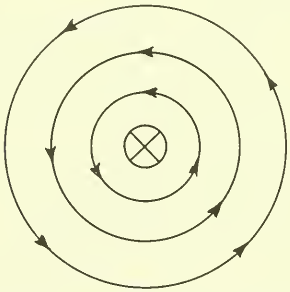


Figure 5-21.—Magnetic lines of force (left hand rule).

The calculus allows us to express this event quite simply. The equation is $E = -\frac{d\phi}{dt}$. This equation states that the induced emf is dependent on the rate of change of the magnetic lines of force. The negative sign indicates that the induced emf produces an effect opposite to the initial change of magnetic lines of force. E is in volts when $\frac{d\phi}{dt}$ is in webers/sec.

Example 5-36.

A conductor is carrying a steady current when the switch to the circuit is suddenly opened. The magnetic flux (Φ) drops from 2.5×10^6 maxwells to 0 maxwells in .05 seconds. Assuming it changes in a uniform manner, calculate the magnitude of the induced emf.

(A) If the flux changes in a uniform manner,

$$\frac{\Delta\phi}{\Delta t} = \frac{d\phi}{dt} = \frac{0 - 2.5 \times 10^6 \text{ maxwells}}{.05 \text{ seconds}}$$

$$\frac{d\phi}{dt} = -0.5 \times 10^8 \text{ maxwells/sec}$$

We have 10^8 maxwells for each weber of flux so that $\frac{d\phi}{dt} = -0.5$ webers/sec.

(B) Since $E = -\frac{d\phi}{dt}$, we have

$$E = +5 \text{ webers/sec} = 5 \text{ volts.}$$

The previous calculations were for a single conductor. Let us carry this a bit further to see what would happen if we were to consider an air core coil with N turns. We will then find that the total number of lines of magnetic force (flux $= \phi$) would be $N\phi$ where N is the number of turns and ϕ is the flux associated with each turn. We would also find that this TOTAL number of flux linkages is proportional to the current in the coil. $N\phi$ is proportional to i. Now to equate the two we must add a constant of proportionality which we will call L. We have then $N\phi = Li$ and we call L the INDUCTANCE of the coil.

We already know that $E = -N\frac{d\phi}{dt}$. If we now differentiate the equation $N\phi = Li$ with respect to time, we get $N\frac{d\phi}{dt} = L\frac{di}{dt}$. Since N and L are constant for any coil.

$$E = -N\frac{d\phi}{dt} = -L\frac{di}{dt}$$

The induced emf in a coil is equal to the inductance of the coil times the rate of change of the current. Again the negative sign tells us that the induced emf opposes the changing current. E will be given in volts when L is in henrys and $\frac{di}{dt}$ is in amperes/sec.

Example 5–37.

An inductance coil of 3 henrys has a uniformly changing current that varies from 5 amperes to 1 ampere in 2 seconds. Calculate the induced emf.

(A) Since the current is changing uniformly,

$$\frac{\Delta i}{\Delta t}=\frac{di}{dt}=\frac{i_f-i_i}{2}=\frac{1-5}{2}=-2 \text{ amp/sec.}$$

(B) $E=-L\dfrac{di}{dt}=-(3)(-2)=6$ volts.

The inductance that we have been discussing so far is called SELF-INDUCTANCE. We will now consider MUTUAL INDUCTANCE. Mutual inductance occurs between two coils when they are arranged so that the flux from one coil links the other. When the flux in the first coil changes, it cuts across a number of the turns in the other coil and this action produces an induced emf in the second coil (as well as the self-induced emf in the first coil). We will use figure 5–22 to illustrate mutual inductance.

With the circuit shown in the figure there is a steady state value for the current through inductor L_1. The voltmeter associated with L_2 WILL NOT deflect. If we change the potentiometer setting to C', the voltage drop across L_1 will decrease and thus will cause a reduction in the current and flux through L_1. Since some of the flux from L_1 is linked with L_2, it will cause a reduction of flux about L_2. An induced emf will be produced across L_2 in such a manner as to oppose this reduction in flux. The induced emf in L_2 created by the reduction in current in L_1 will be indicated on the voltmeter. When the current again reaches a steady state value, the voltmeter will return to zero.

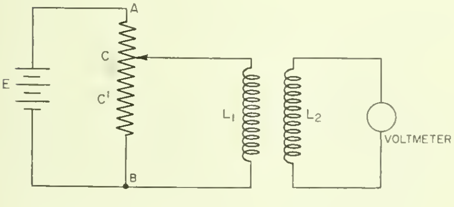


Figure 5–22.—Mutual inductance between L_1 and L_2.

The magnitude of the induced emf can be determined from $E=-M\dfrac{di}{dt}$ where M is the mutual inductance between L_1 and L_2 and $\dfrac{di}{dt}$ is the rate of change of the current in L_1.

The discussion of derivatives is continued to another circuit element, the capacitor.

The charge on a capacitor is given by the expression $Q=CV$ where Q is the charge, C the capacitance, and V is the voltage across the capacitor. Assuming the capacitance is constant, we may differentiate the expression with respect to time.

$$Q=CV$$

$$\frac{dQ}{dt}=C\frac{dv}{dt}$$

We know by definition that the current in a conductor is the flow of electrons past a point in the conductor per unit of time.

$$\therefore \qquad \frac{dQ}{dt}=i=C\frac{dv}{dt}$$

This expression tells us that the current in a circuit containing a capacitor is equal to the capacitance of the capacitor times the rate of change of the voltage across the capacitor. The current is in amperes when the capacitance is in farads and the rate of change of voltage is in volts per second.

In figure 5–23 we have an application of this formula. The capacitor and central zero ammeter are connected to a potentiometer, P_1, on one side and a center tapped resistor, R, on the other.

As we vary the potentiometer setting from a to b or b to a, we find that the current in the ammeter is dependent on the speed at which this change is made. The faster we move the potentiometer setting from a to b, the greater the current produced. We expect this since the equation, $i=C\dfrac{dv}{dt}$, clearly indicates the magnitude of the current depends on the rate of change of voltage applied across the capacitor.

Example 5–38.

Suppose a change in potentiometer setting from a to b covers a change in voltage of 40 volts. Determine the current indicated on the ammeter if the

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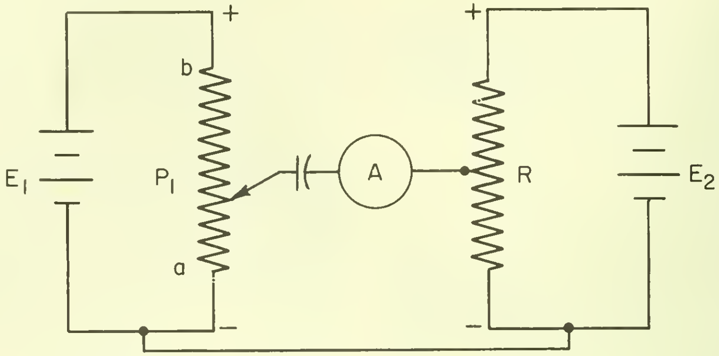

Figure 5-23.—Capacitor current dependent on change of capacitor voltage.

change is made uniformly in (1) 2 seconds, and (2) 0.05 second. The capacitor value is 10 microfarads.

(1) A. In a uniform change,

$$\frac{\Delta v}{\Delta t}=\frac{dv}{dt}=\frac{40}{2}=20 \text{ volts/sec}$$

B. $i=C\dfrac{dv}{dt}=(10\times 10^{-6} \text{ farads})(20 \text{ volts/sec})$

$i=2\times 10^{-4}$ amperes

(2) A. Δt is now 0.05 second.

$$\frac{\Delta v}{\Delta t}=\frac{dv}{dt}=\frac{40}{0.05}=800 \text{ volts/sec}$$

B. $i=C\dfrac{dv}{dt}=(10\times 10^{-6} \text{ farads})(800 \text{ volts/sec})$

$i=8\times 10^{-3}$ amperes

5-20. Differentials

Before we attempt to define the differential of a function, let us recall that we defined the derivative of $y=f(x)$ as

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

In words, we say that the derivative of $y=f(x)$ is the limit of a ratio Δy to Δx as Δx approaches zero in the limit.

Once we differentiated the function to obtain the derivative, WE NEVER spoke of dy alone or dx alone. However, when we speak of differentials WE DO speak of dy and dx as separate parts of the derivative $\dfrac{dy}{dx}$.

A close look at figure 5-24 should help to make clear exactly what we mean by the differential of a function.

Notice as we move along the curve $y=f(x)$ from A to B, x takes on an INCREMENT Δx and y takes on an INCREMENT Δy.

We have drawn the tangent to $y=f(x)$ at point A. Recall that the slope of the tangent line at a point is equal to the slope of the function at that point.

$$\therefore \frac{dy}{dx}=\tan\alpha=\frac{dy}{\Delta x}=f'(x)$$

From the above relationship we may say in the case of the INDEPENDENT VARIABLE, x, the INCREMENT, Δx, of the independent variable is equal to the DIFFERENTIAL, dx, of the independent variable. Since $\dfrac{dy}{dx}=\dfrac{dy}{\Delta x}$, we have $\Delta x=dx$.

This equation defines what we mean by the differential of the independent variable.

We now have the equation $\dfrac{dy}{dx}=f'(x)$, but now we treat the $\dfrac{dy}{dx}$ portion as a ratio of the two parts dy and dx. (These two parts are called the differentials, dy and dx.)

Let us take the equation $\dfrac{dy}{dx}=f'(x)$ and multiply both sides by dx;

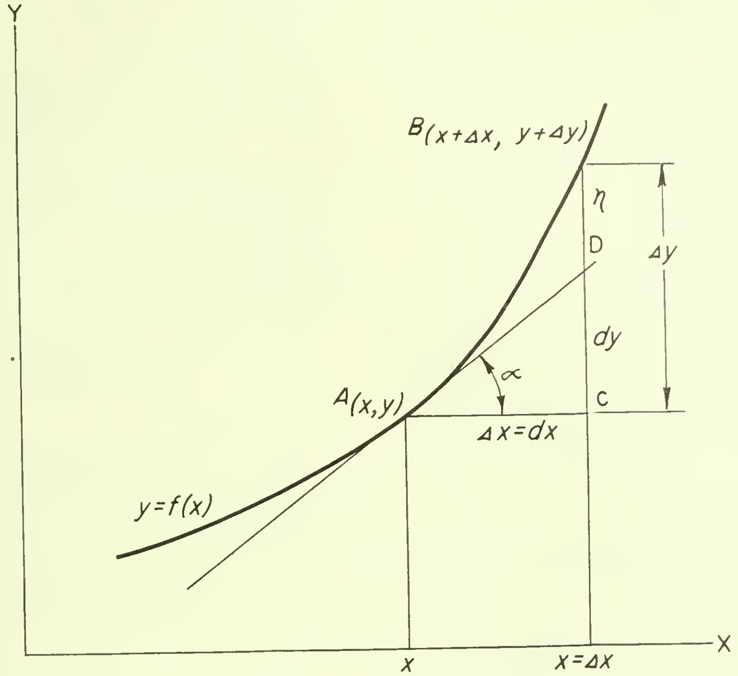

Figure 5–24.—Difference between dy and Δy.

$$\frac{dy}{dx}\cdot dx=f'(x)dx$$

$$\therefore\ dy=f'(x)dx$$

This equation defines the DIFFERENTIAL OF THE DEPENDENT VARIABLE. Normally we call dy the differential of the function.

It should be evident that the differential of the function, dy, is NOT EQUAL to the increment of the function, Δy (fig. 5–24). Notice that $\Delta y=dy+\eta$ so that $\eta=\Delta y-dy$. We see the differential dy is a good approximation to Δy only if η is small. dy will only approach Δy if η approaches zero, that is, only if η is an INFINITESIMAL. This will occur only if Δx is chosen small.

The following statements summarize what we have said about differentials.

(1) The increment and differential of the independent variable ARE EQUAL.

$$\Delta x=dx$$

(2) The differential of a function is equal to the product of the derivative of the function and the differential of the independent variable.

$$dy=F'(x)dx$$

(3) The increment of the function is NOT EQUAL to the differential of the function.

$$\Delta y=dy+\eta$$

(4) By choosing Δx small enough, we may make η small so that dy is a good approximation to Δy.

The following formulas for differentials are given without proof and correspond to the formulas previously given for the derivatives of these similar forms.

1. $d(c)=0$
2. $d(x^n)=nx^{n-1}dx$
3. $d(u+v)=du+dv$
4. $d(cv)=cdv$
5. $d(uv)=vdu+udv$
6. $d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}$
7. $d(u^n)=nu^{n-1}du$

5-21. Problems Involving Differentials

Calculating the increment of a function is normally quite long and involved. By using the differential to approximate the increment, the problem solving time is greatly reduced. In most cases the error that arises may be neglected. A few examples will best illustrate the general procedure.

Example 5-39.

Find the approximate area of one surface of a ring of 9 in. inside diameter and 9.25 in. outside diameter (fig. 5-25).

(A) We could of course simply subtract the area of circle 1 from the area of circle 2, but first we choose to do the problem by differentials as we need only an approximate answer.

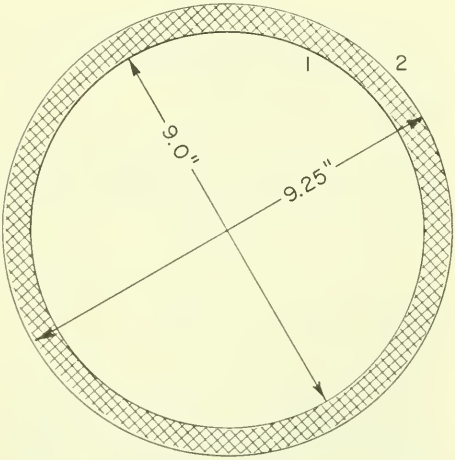

Figure 5-25.—Find the cross hatched area.

(B) $A=\pi r^2$

$dA=2\pi r dr$ (We will call $dA=A_R$)

$$\text{let } r=\frac{9}{2}\text{ in.}=4.5\text{ in. and } dr=\frac{9.25}{2}-\frac{9}{2}\text{ in.}$$

$$dr=0.125\text{ in.}$$

$$\therefore \qquad A_R=(2)(3.14)(4.5)(0.125)=3.53\text{ in}^2$$

(C) We will now work the problem by subtracting the areas of circles 1 and 2 in order to see what error we introduced.

(D) $A_1=\pi r_1^2=\pi\frac{D_1^2}{4}=(3.14)\left(\frac{9}{4}\right)^2$

$A_1=63.5\text{ in}^2$

$A_2=\pi r_2^2=\pi\frac{D_2^2}{4}=(3.14)\frac{(9.25)^2}{4}$

$A_2=67.1\text{ in}^2$

$A_R=A_2-A_1=67.1-63.5=3.6\text{ in}^2$

(E) The error introduced is equal to the difference in the more correct result (3.6 in^2) and the approximate result (3.53 in^2).

Error $=dA=3.6\text{ in}^2-3.53\text{ in}^2=.07\text{ in}^2$

(Here dA is the difference in areas.)

(F) The relative error is the ratio of the error to the more correct answer.

$$\text{Relative error}=\frac{dA}{A}=\frac{.07\text{ in}^2}{3.6\text{ in}^2}=.0195$$

(G) The percentage error is the relative error multiplied by 100.

$$\text{Percentage (\%) error}=100\frac{dA}{A}=(0.0195)(100)=1.95\%$$

Example 5-40.

The expression for power lost in a resistor is $P=i^2R$ where P is the power lost and i is the current passing through the resistance R. If we can measure the current within $\pm 3\%$ tolerances, what are the corresponding tolerances in the power calculations?

(A) The $\pm 3\%$ tolerances in the current measurements tell us that

$$100\frac{di}{i}=\pm 3\%. \qquad \frac{di}{i}=\pm .03$$

(B) $P=i^2R$

$$dP=2idiR=2iRdi$$

But $di = \pm 0.03i$

$$\therefore dP = 2iR(\pm 0.03i) = \pm 0.06i^2R$$

(C) We are looking for the % error (or tolerances) in the power calculations.

$$\frac{dP}{P} = \pm\frac{0.06i^2R}{i^2R} = \pm 0.06$$

$$\therefore 100\frac{dP}{P} = (100)(\pm .06) = \pm 6\%$$

An error of ±3% in reading the current values makes it impossible for us to calculate the corresponding power values any closer than ±6%.

Example 5–41.

Find the approximate cube root of 340 by using differentials.

(A) We form an expression for calculating the cube root of any number: $y = \sqrt[3]{x} = x^{1/3}$

(B) We now find which number is the nearest perfect cube to 340. In our case $7^3 = 343$ and we let $x = 343$.

$\therefore dx = -3$ since our number differs from 343 by -3.

(C) Take the differential of y;

$$dy = \frac{1}{3}x^{2/3}\,dx = \frac{dx}{3x^{2/3}}$$

and $dx = -3$, $x = 343$

$$\therefore dy = \left(\frac{1}{3}\right)\frac{-3}{343^{2/3}} = -\frac{1}{49} = -.0204$$

(D) The cube root of 340 is then $\sqrt[3]{343} + dy = 7 - .0204 = 6.9796$ ($+dy$ is a negative number).

(E) Six place cube root tables give $\sqrt[3]{340}$ as 6.979532. Our approximation is good to the fourth decimal place.

Exercise 5–6.

1. We are interested in calculating the volume of a certain sphere. Unknowingly the radius is recorded with a relative error of +.04, what % error will result in the volume if the radius is given as 6 feet?

2. A conductor with a 3.5-inch diameter is insulated with a layer of insulation .025 in. thick. What is the approximate cross-sectional area of this insulation?

CHAPTER 6

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS

6–1. Introduction

In this chapter we continue the development of differentiation. We increase the scope to include those functions known as TRANSCENDENTAL FUNCTIONS. A transcendental function is simply a nonalgebraic function.

We will investigate the trigonometric functions and their inverse functions, and the exponential functions and their inverse functions.

A review of the basic ideas related to trigonometry, exponents, and logarithms should be undertaken at this point unless the reader is already familiar with these subjects.

6–2. Trigonometric Functions

The trigonometric functions include sine, cosine, tangent, secant, cosecant, and cotangent. The relationships that define these functions, as well as several identities between them are given in chapter 1.

Figure 6–1 contains values for the trigonometric functions through 360° of arc at 30° intervals.

The corresponding angle is given in radian measurements ($180° = 3.1416$ radians $= \pi$ radians). The values are correct to the second decimal place where shown. The squares with ∞ indicate this function is APPROACHING ∞ for these angular values and DOES NOT mean that the function is equal to infinity.

Graphs of all the trigonometric functions are given in figures 6–2 through 6–7. We shall follow the graphing of csc θ as an example of the procedure followed for the remainder of the functions.

Example 6–1.

Graph csc θ through 360° (fig. 6–3).

A. We first note that

$$\csc \theta = \frac{1}{\sin \theta} \cdot$$

B. Construct a table giving values for θ, sin θ, and csc θ.

C. Construct the coordinate axes with csc θ on the vertical axis and θ (radians) plotted on the horizontal axis. Choose some convenient scale for the horizontal axis and divide it into four equal parts each equal to $\frac{\pi}{2}$ radians. Approximate where one radian length would fall in the horizontal axis and use this length for the units in the vertical axis.

D. We know that from 0 to π, the sign for sin θ is positive. Therefore, the sign for csc θ is positive and when $\theta = 0$, csc θ is approaching positive infinity. At $\frac{\pi}{2}$ the csc θ is 1 and we sketch the curve as shown in figure 6–3 between 0 and $\frac{\pi}{2}\cdot$ As θ goes beyond $\frac{\pi}{2}$ and approaches π, we see that csc θ approaches positive infinity again.

TABLE 6–1.—VALUES OF θ, SIN θ, AND CSC θ.

θ (degrees)	θ (radians)	sin θ	csc θ
0	0	0	$+\infty$
90°	$\frac{\pi}{2}$	1	1
180°	π	0	$\pm\infty$
270°	$\frac{3\pi}{2}$	-1	-1
360°	2π	0	$-\infty$

Once we pass beyond π to values of θ in which $\pi < \theta < 2\pi$, sin θ is negative and hence csc θ is negative.

It must be true that for values of θ just to the left of π, csc θ approaches $+\infty$, and for values of

θ (DEGREES)	0°	30	60	90	120	150	180	210	240	270	300	330	360
θ (RADIANS)	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
SIN θ	0	.5	.87	1	.87	.5	0	-.5	-.87	-1	-.87	-.5	0
CSC θ	∞	2	1.2	1	1.2	2	∞	-2	-1.2	-1	-1.2	-2	∞
COS θ	1	87	.5	0	-.5	-.87	-1	-.87	-.5	0	.5	.87	1
SEC θ	1	1.2	2	∞	-2	-1.2	-1	-1.2	-2	-∞	2	1.2	1
TAN θ	0	.58	1.7	∞	-1.7	-.58	0	.58	1.7	∞	-1.7	-.58	0
COT θ	∞	1.7	.58	0	-.58	-1.7	-∞	1.7	.58	0	-.58	-1.7	-∞

Figure 6-1.—Table of trigonometric values correct to two decimal places.

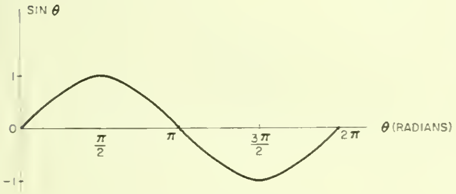

Figure 6-2.—Graph of sin θ.

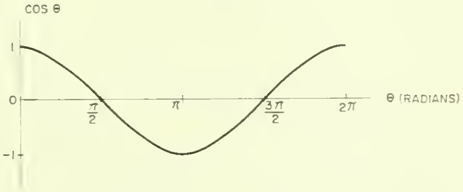

Figure 6-4.—Graph of cos θ.

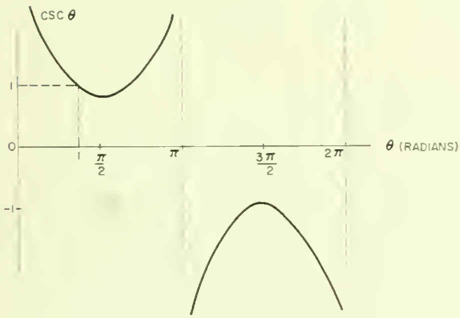

Figure 6-3.—Graph of csc θ.

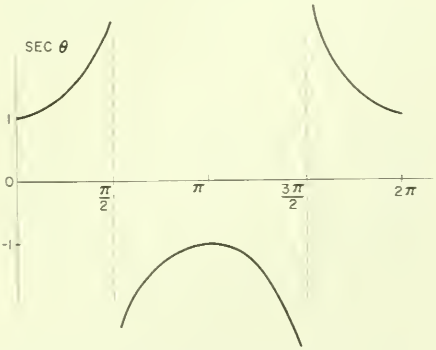

Figure 6-5.—Graph of sec θ.

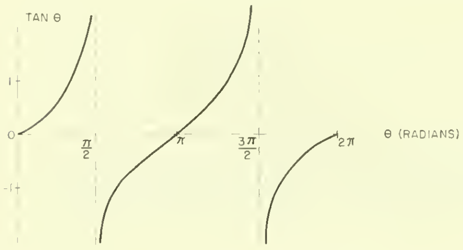

Figure 6–6.—Graph of tan θ.

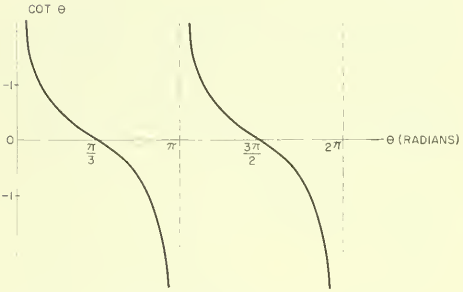

Figure 6–7.—Graph of cot θ.

θ just to the right of π, csc θ approaches $-\infty$. These facts are shown on the graph.

The remainder of the graph should now be clear.

6–3. Inverse Trigonometric Functions

As we saw in section 5–12, with every function there is a related inverse function. In the case of trigonometric functions, the inverse function takes on an important role.

The trigonometric function $x=\tan y$ has as its inverse function $y=\text{arc tan } x$ or $y=\tan^{-1} x$. Both of these latter relationships are read, "$y=\text{arc tan } x$" or, "y is the angle whose tangent is x," i.e., $\tan y=x$.

We must ensure that our notation for the inverse trigonometric functions is clear. If we mean the inverse function, the -1 will be placed a little above and to the right of the name of the type of inverse trigonometric function as in $y=\sin^{-1} x$. If we wish to indicate that the trigonometric function is to be raised to the exponent -1, we will place parenthesis about the function and put the -1 above and to the right of the parenthesis as in $y=(\sin x)^{-1}$.

The inverse trigonometric functions are shown in figures 6–8 through 6–13.

Example 6–2.

Graph the inverse trigonometric function $y=\text{arc tan } x$ (fig. 6–12).

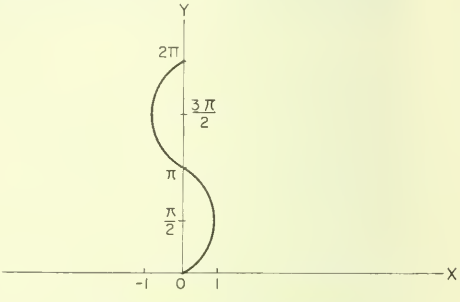

Figure 6–8.—Graph of $y=\text{arc sin } x$.

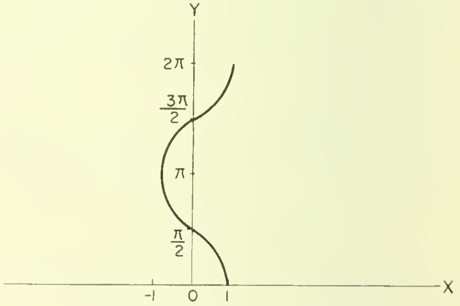

Figure 6–9.—Graph of $y=\text{arc cos } x$.

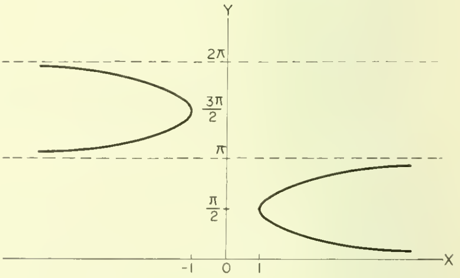

Figure 6–10.—Graph of $y=\text{arc csc } x$.

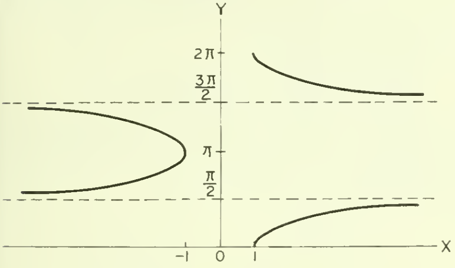

Figure 6–11.—Graph of y=arc sec x.

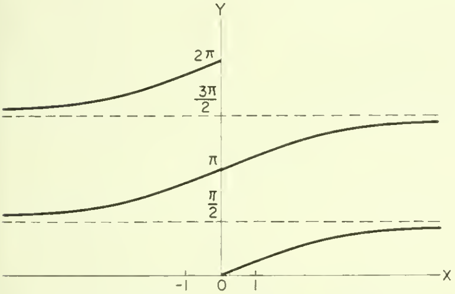

Figure 6–12.—Graph of y=arc tan x.

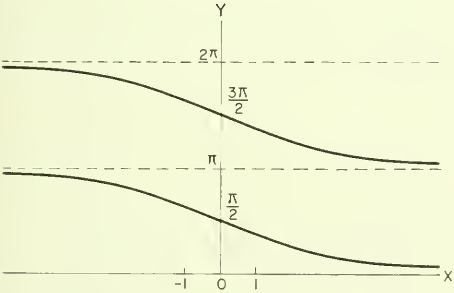

Figure 6–13.—Graph of y=arc cot x.

A. The above function is the same as writing, "y is the angle whose tangent is x." A triangle illustrating this fact is given in figure 6–14.

B. The equivalent relation $x=\tan y$ will be plotted. Choose values for y in radians and determine x.

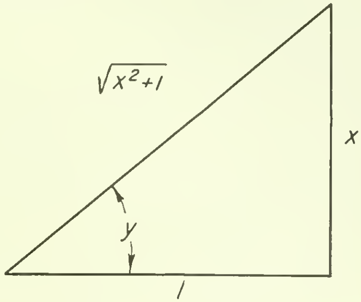

Figure 6–14.—y=arc tan x.

y	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	0	$\pm\infty$	0	$\pm\infty$	0

C. Construct the axis to plot the values given in the table from part B.

D. Plot the values taking care to note the sign of the function tan y in the various intervals. In the interval $0<y<\frac{\pi}{2}$, tan y is positive and x approaches $+\infty$ as y approaches $\frac{\pi}{2}$. In the interval $\frac{\pi}{2}<y<\pi$, tan y is negative so that x approaches 0 from the $-\infty$ direction. This technique is followed throughout the region in which we are interested. The same procedure is used to plot the remaining inverse trigonometric functions.

6–4. The Definition of *e*

An important quantity with which we will be concerned in the remainder of our work is the number e. It is the base of our natural logarithms to be discussed in section 6–6.

We define this constant from a specific limit:

$$e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=2.71828\ldots\ldots$$

The number e, like the number represented by π, is an irrational number since it cannot be represented by any finite quantity of numbers. We choose 2.718 as a working value for e.

6–5. Exponential Functions

The expression $y=a^u$, where u is any function of x and a is any positive constant except one, is called an EXPONENTIAL FUNCTION of x. In many of our problems "a" will be equal to e where e is the quantity previously defined in section 6–4. The curves of the exponential functions $y=10^x$ and $y=e^x$ are plotted in figure 6–15.

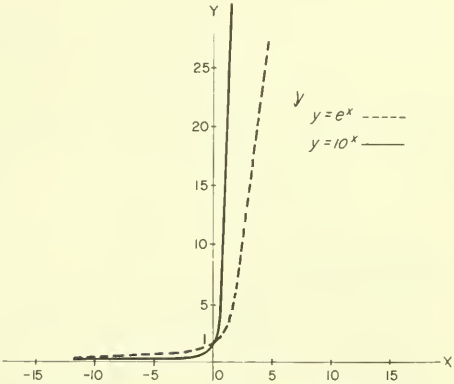

Figure 6–15.—Graph of an exponential function of x.

6–6. Logarithmic Functions

In the expression $y=a^u$, u IS THE LOGARITHM OF y TO THE BASE a. That is, if we solve the equation $y=a^u$ for u, it will be equal to the logarithm of y to the base a;

$$u=\log_a y.$$

The expression solved for u is the inverse function of the original exponential function. We may say that THE LOGARITHMIC FUNCTIONS ARE THE INVERSE OF THE EXPONENTIAL FUNCTIONS.

The restrictions which we put on "a" in the previous section are important and must be remembered. In most logarithmic work, the base a is restricted to either 10 or e. Logarithms to the base 10 are called common logarithms, and logarithms to the base e are called natural or Napierian logarithms. The notation which we will use for these two bases will be;

Common logarithm: $\log_{10} N$ or $\log N$

Natural logarithm: $\log_e N$ or $\ln N$

Suppose we ask what is $\log_{10} 0$ or $\log_e 0$? We can immediately determine these answers from the equations $y=10^x$ and $y=e^x$, for what we are asking is what value of x will give us $y=0$.

$$\begin{array}{ll} y=10^x & y=e^x \\ 0=10^x & 0=e^x \\ \therefore\ x=-\infty \text{ for} & \therefore\ x=-\infty \text{ for} \\ 0=10^{-\infty}=\dfrac{1}{10^{\infty}} & 0=e^{-\infty}=\dfrac{1}{e^{\infty}} \end{array}$$

Therefore, we conclude that the logarithm of zero to ANY BASE is $-\infty$.

We follow similar steps to determine $\log_{10} 1$ or $\log_e 1$. We are asking: If $y=1$, what must x be?

$$\begin{array}{ll} y=10^x & y=e^x \\ 1=10^x & 1=e^x \\ \therefore\ x=0 & \therefore\ x=0 \end{array}$$

Our conclusion is that the logarithm of one to ANY BASE is zero since any base raised to the power of zero is one.

We now wish to determine what numbers have a logarithm equal to 1 in both the common and natural logarithmic systems. We are asking: If $x=1$, what is y?

$$\begin{array}{ll} y=10^1 & y=e^1 \\ \therefore\ y=10 & \therefore\ y=e \end{array}$$

We conclude that the $\log_a a=1$, i.e., $\log_{10} 10=1$ and $\log_e e=1$.

Figure 6–16 is a graph of the equation $y=\ln x$.

6–7 Formulas for Important Derivatives

We list here for easy reference the list of important derivatives of transcendental functions. The derivations follow. This list, plus the list included in chapter 5, contains all the derivatives we will use. The symbol u denotes a function of x.

11. $\dfrac{d(\sin u)}{dx}=\cos u\dfrac{du}{dx}$

12. $\dfrac{d(\cos u)}{dx}=-\sin u\dfrac{du}{dx}$

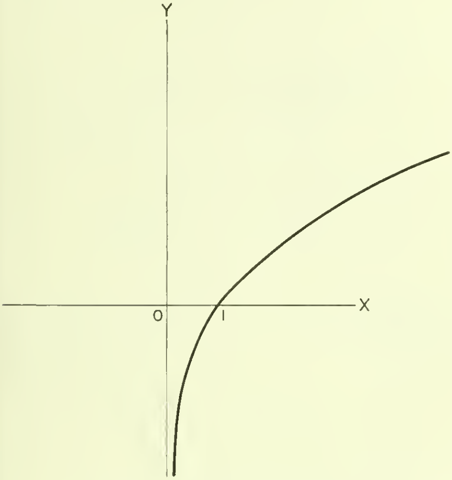

Figure 6–16.—Graph of $y=\ln x$.

13. $\dfrac{d(\tan u)}{dx}=\sec^2 u\dfrac{du}{dx}$

14. $\dfrac{d}{dx}(\cot u)=-\csc^2 u\dfrac{du}{dx}$

15. $\dfrac{d}{dx}(\sec u)=\sec u\tan u\dfrac{du}{dx}$

16. $\dfrac{d}{dx}(\csc u)=-\csc u\cot u\dfrac{du}{dx}$

17. $\dfrac{d}{dx}(\text{arc}\sin u)=\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$

18. $\dfrac{d}{dx}(\text{arc}\cos u)=-\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$

19. $\dfrac{d}{dx}(\text{arc}\tan u)=\dfrac{1}{1+u^2}\dfrac{du}{dx}$

20. $\dfrac{d}{dx}(\text{arc}\cot u)=-\dfrac{1}{1+u^2}\dfrac{du}{dx}$

21. $\dfrac{d}{dx}(\text{arc}\sec u)=\dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$

22. $\dfrac{d}{dx}(\text{arc}\csc u)=-\dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$

23. $\dfrac{d}{dx}(\log_a u)=\dfrac{\log_a e}{u}\dfrac{du}{dx}$

24. $\dfrac{d(\ln u)}{dx}=\dfrac{1}{u}\dfrac{du}{dx}$

25. $\dfrac{d}{dx}(a^u)=a^u\ln a\dfrac{du}{dx}$

26. $\dfrac{d}{dx}(e^u)=e^u\dfrac{du}{dx}$

6–8. Determination of an Important Limit

Before we begin deriving the formulas given in the previous section, it is necessary for us to determine the following limit: $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}$. This limit occurs in deriving the derivative of $\sin\theta$ and must be evaluated at this point.

Notice that we obtain the indeterminant expression $\dfrac{0}{0}$ by direct substitution and must, therefore, seek other means. We will resort to construction and trigonometric relations as shown in figure 6–17. In this figure angles ϕ and δ are right angles.

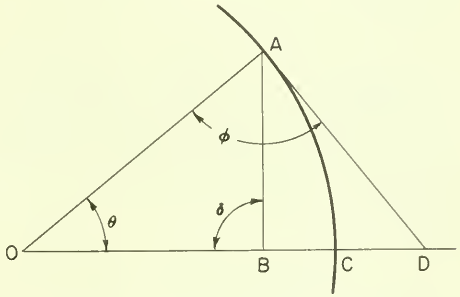

Figure 6–17.—Construction for proving $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$.

1. $\sin\theta=\dfrac{AB}{AO}$, $(\Delta OBA)\ \therefore AB=AO\sin\theta$

2. $\tan\theta=\dfrac{AD}{AO}$, $(\Delta ODA)\ \therefore AD=AO\tan\theta$

3. The arc AC = radius of the arc multiplied by the angle θ swept out in producing the arc, measured in radians.

$$\therefore AC=(AO)\theta$$

4. It is true that $AB\leq AC\leq AD$

5. Substitute 1, 2, and 3 in 4.

$$\therefore AO\sin\theta\leq(AO)\theta\leq AO\tan\theta$$

6. Cancel the common factor AO:

$$\sin\theta\leq\theta\leq\tan\theta.$$

7. $\tan\theta=\dfrac{\sin\theta}{\cos\theta}\ \therefore \sin\theta\leq\theta\leq\dfrac{\sin\theta}{\cos\theta}$

8. Divide through by $\sin\theta$;

$$1 \leqslant \frac{\theta}{\sin\theta} \leqslant \frac{1}{\cos\theta}.$$

9. Invert and reverse inequality signs;

$$1 \geqslant \frac{\sin\theta}{\theta} \geqslant \cos\theta.$$

10. If we allow θ to approach zero and investigate the expression in 9, we see that $\cos\theta$ approaches 1. We also see that 1 is equal to or greater than $\frac{\sin\theta}{\theta}$ and $\frac{\sin\theta}{\theta}$ is equal to or greater than $\cos\theta$ which is approaching 1. That is, $1 \geqslant \frac{\sin\theta}{\theta} \geqslant 1$. Therefore as $\theta \to 0$, $\frac{\sin\theta}{\theta}=1$ for it is between 1 and 1 which can only be 1.

11. Finally, $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1.$

6-9. Derivations of Trigonometric Differentiation Formulas

We will follow the familiar steps of the delta process in deriving the basic formulas. The steps are outlined in chapter 5.

We will first differentiate $\sin u$ where u is a function of x.

11. A. $y=\sin u$

B. $y+\Delta y=\sin(u+\Delta u)$

C. $\Delta y=\sin(u+\Delta u)-\sin u$

D. $\frac{\Delta y}{\Delta u}=\frac{\sin(u+\Delta u)-\sin u}{\Delta u}$, but

$\sin(A+B)=\sin A\cos B+\sin B\cos A$ from chapter one. Let $u=A$ and $\Delta u=B$ so that $\sin(u+\Delta u)=\sin u\cos\Delta u+\sin\Delta u\cos u$.

E. $\therefore \frac{\Delta y}{\Delta u}=\frac{\sin u\cos\Delta u+\sin\Delta u\cos u-\sin u}{\Delta u}$

Regrouping: $\frac{\Delta y}{\Delta u}=\frac{\sin u(\cos\Delta u-1)}{\Delta u}+\frac{\sin\Delta u\cos u}{\Delta u}$

F. $$\frac{dy}{du}=\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}=\lim_{\Delta u\to 0}\frac{\sin u(\cos\Delta u-1)}{\Delta u}+\lim_{\Delta u\to 0}\frac{\sin\Delta u\cos u}{\Delta u}$$

G. $\lim\limits_{\Delta u\to 0}\frac{\sin u(\cos\Delta u-1)}{\Delta u}=0,$

since

$$\lim_{\Delta u\to 0}\frac{\cos\Delta u-1}{\Delta u}\cdot\frac{\cos\Delta u+1}{\cos\Delta u+1}=\lim_{\Delta u\to 0}\frac{-\sin\Delta u\sin\Delta u}{(\cos\Delta u+1)\Delta u}=-\frac{0}{2}\cdot 1$$

H. $\lim\limits_{\Delta u\to 0}\frac{\cos u\sin\Delta u}{\Delta u}=\cos u$ since

$\lim\limits_{\Delta u\to 0}\frac{\sin\Delta u}{\Delta u}=1$ from section 6–8.

I. Therefore, in F, $\frac{dy}{du}=\cos u$

J. If $y=\sin u$, $\frac{dy}{du}=\cos u$, what is $\frac{dy}{dx}$?

K. Recall that if $y=f(u)$, and u is $f(x)$ then $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$.

L. Therefore, $\frac{dy}{dx}=\frac{d}{dx}(\sin u)=\cos u\frac{du}{dx}$.

12. A. $y=\cos u$

B. $y+\Delta y=\cos(u+\Delta u)$

C. $\Delta y=\cos(u+\Delta u)-\cos u$

D. $\frac{\Delta y}{\Delta u}=\frac{\cos(u+\Delta u)-\cos u}{\Delta u}$, but $\cos(A+B)$

$=\cos A\cos B-\sin A\sin B$ from chapter 1. Let $u=A$ and $\Delta u=B$ so that $\cos(u+\Delta u)=\cos u\cos\Delta u-\sin u\sin\Delta u$.

E. $\therefore \frac{\Delta y}{\Delta u}=\frac{\cos u\cos\Delta u-\sin u\sin\Delta u-\cos u}{\Delta u}$

Regrouping: $\frac{\Delta y}{\Delta u}=\frac{\cos u(\cos\Delta u-1)}{\Delta u}-\frac{\sin u\sin\Delta u}{\Delta u}$

F. $$\frac{dy}{du}=\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}=\lim_{\Delta u\to 0}\frac{\cos u(\cos\Delta u-1)}{\Delta u}-\lim_{\Delta u\to 0}\frac{\sin u\sin\Delta u}{\Delta u}$$

G. $\lim\limits_{\Delta u\to 0}\frac{\cos u(\cos\Delta u-1)}{\Delta u}=0$

since $\lim\limits_{\Delta u\to 0}\frac{\cos\Delta u-1}{\Delta u}=0$ from previous derivation of #11.

H. $-\lim\limits_{\Delta u\to 0}\frac{\sin u\sin\Delta u}{\Delta u}=-\sin u$, since

$\lim\limits_{\Delta u\to 0}\frac{\sin\Delta u}{\Delta u}=1$ as before.

I. Therefore, in F, $\frac{dy}{du}=-\sin u$

J. If $y=\cos u$, $\frac{dy}{du}=-\sin u$ and we need $\frac{dy}{dx}$.

K. Again $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$. Therefore,

$$\frac{dy}{dx}=\frac{d(\cos u)}{dx}=-\sin u\frac{du}{dx}.$$

13. A. $y=\tan u=\frac{\sin u}{\cos u}$

B. Rather than apply the delta process to this ratio, we choose to treat it as a form of formula 6, in chapter 5. Substituting $\sin u$ for u, $\cos u$ for v, and du for dx in the expression

$$\frac{d\left(\frac{u}{v}\right)}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}:$$

$$\frac{dy}{du}=\frac{d\left(\frac{\sin u}{\cos u}\right)}{du}=\frac{\cos u\frac{d(\sin u)}{du}-\sin u\frac{d(\cos u)}{du}}{\cos^2 u}$$

Substituting from (11) $\frac{d(\sin u)}{du}=\cos u$ and from (12) $\frac{d(\cos u)}{du}=-\sin u$

$$\frac{dy}{du}=\frac{\cos^2 u+\sin^2 u}{\cos^2 u}=1+\tan^2 u$$

Therefore, $\frac{dy}{du}=1+\tan^2 u=\sec^2 u$ from chapter 1.

C. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ and $\frac{dy}{dx}=\frac{d(\tan u)}{dx}=\sec^2 u\frac{du}{dx}$

14. A. $y=\cot u=\frac{\cos u}{\sin u}$

B. Apply the quotient rule (formula 6) from chapter 5. Substituting $\cos u$ for u, $\sin u$ for v, and du for dx in the expression

$$\frac{d\left(\frac{u}{v}\right)}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{du}=\frac{\sin u\frac{d(\cos u)}{du}-\cos u\frac{d(\sin u)}{du}}{\sin^2 u}$$

Substituting from (12) $\frac{d(\cos u)}{du}=-\sin u$ and from (11) $\frac{d(\sin u)}{du}=\cos u$

$$\frac{dy}{du}=\frac{-\sin^2 u-\cos^2 u}{\sin^2 u}=\frac{-(\sin^2 u+\cos^2 u)}{\sin^2 u}$$

Substituting 1 for $\sin^2 u+\cos^2 u$ and $\csc u$ for $\frac{1}{\sin u}$

$$\frac{dy}{du}=-\frac{1}{\sin^2 u}=-\csc^2 u.$$

C. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ and $\frac{dy}{dx}=\frac{d(\cot u)}{dx}=-\csc^2 u\frac{du}{dx}$

15. A. $y=\sec u=\frac{1}{\cos u}$

B. We again apply the quotient formula (6) for differentiation from chapter 5. Substituting 1 for u, $\cos u$ for v and du for dx in the expression

$$\frac{d\left(\frac{u}{v}\right)}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{du}=\frac{\cos u\frac{d(1)}{du}-(1)\frac{d(\cos u)}{du}}{\cos^2 u}$$

Substituting from (1) ch 5 $\frac{d(1)}{du}=0$ and from (12) ch 6 $\frac{d(\cos u)}{du}=-\sin u$

$$\frac{dy}{du}=\frac{\sin u}{\cos^2 u}=\frac{1}{\cos u}\frac{\sin u}{\cos u}=\sec u\tan u$$

C. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ and $\frac{dy}{dx}=\frac{d(\sec u)}{dx}$

$$=\sec u\tan u\frac{du}{dx}$$

16. A. $y=\csc u=\frac{1}{\sin u}$

Substitute 1 for u, $\sin u$ for v and du for dx in the expression $\frac{d\left(\frac{u}{v}\right)}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

B. $\dfrac{dy}{du}=\dfrac{\sin u\dfrac{d\,(1)}{du}-(1)\dfrac{d\,(\sin u)}{du}}{\sin^2 u}$ and

Substituting from (1) ch 5 $\dfrac{d(1)}{du}=0$ and from (11) ch 6

$$d\frac{(\sin u)}{du}=\cos u$$

$$\frac{dy}{du}=-\frac{\cos u}{\sin^2 u}=-\frac{1}{\sin u}\,\frac{\cos u}{\sin u}=-\csc u\cot u$$

C. $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$ and $\dfrac{dy}{dx}=\dfrac{d\,(\csc u)}{dx}$

$$=-\csc u\cot u\frac{du}{dx}$$

Example 6–3.

Calculate the first derivative of the following expressions:

1. $y=\sin 2x$. We have $y=\sin u$ with $u=2x$. from (11) $\dfrac{dy}{dx}=\cos 2x\dfrac{d(2x)}{dx}=2\cos 2x$. Since $\dfrac{d(2x)}{dx}=2$ from (10) ch 5.

2. $y=(\cos x^2)^2$. We may visualize this expression as $y=u^n$ so that $\dfrac{dy}{dx}=nu^{n-1}\dfrac{du}{dx}$ from (7) ch 5. As u is $\cos x^2$, $\dfrac{du}{dx}$ must be $\dfrac{d\,(\cos x^2)}{dx}=-2x\sin x^2$ from (12) and $n=2$ and $n-1=1$.

Therefore, $\dfrac{d(\cos x^2)^2}{dx}=2(\cos x^2)^1(-2x\sin x^2)$ or

$$\frac{dy}{dx}=-4x\cos x^2\sin x^2$$

3. $y=\tan\dfrac{x}{2}$

Applying (13) let $u=\dfrac{x}{2}$

$$\frac{dy}{dx}=\sec^2\frac{x}{2}\,\frac{d\left(\frac{x}{2}\right)}{dx}=\frac{1}{2}\sec^2\frac{x}{2}$$

where

$$\frac{d\left(\frac{x}{2}\right)}{dx}=\frac{1}{2}\qquad\text{from (10) ch 5}$$

4. $y=\cos(x^2-1)\sin 3x$. We treat this as a product and differentiate. We let $u=\cos(x^2-1)$ and $v=\sin 3x$.

If $y=uv$, $\dfrac{dy}{dx}=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$ formula (5) ch 5.

$$\frac{dy}{dx}=\cos(x^2-1)\frac{d(\sin 3x)}{dx}+\sin 3x\frac{d[\cos(x^2-1)]}{dx}\tag{6–1}$$

$$\frac{d\,(\sin 3x)}{dx}=\cos 3x\frac{d(3x)}{dx}\qquad\text{from (11)}$$

$$=3\cos 3x\qquad\text{from (10) ch 5}\tag{6–2}$$

$$\frac{d\,[\cos(x^2-1)]}{dx}=-\sin(x^2-1)\frac{d(x^2-1)}{dx}\ \text{from (12)}\tag{6–3}$$

$$\frac{d(x^2-1)}{dx}=2x\qquad\text{from (4) and (1) ch 5}\tag{6–4}$$

Substituting (6–2), (6–3), and (6–4) in (6–1)

$$\frac{dy}{dx}=\cos(x^2-1)\ 3\cos 3x+(\sin 3x)\,[-\sin(x^2-1)]2x$$

$$=3\cos 3x\cos(x^2-1)-2x\sin 3x\sin(x^2-1)$$

6–10. Derivations of the Inverse Trigonometric Forms

17. A. $y=\sin^{-1}u=\text{arc}\sin u$

These equations tell us y is the angle whose sine is u. Figure 6–18 illustrates the right triangle with the angle y.

B. $u=\sin y$

C. $\dfrac{d\,(u)}{du}=\dfrac{d(\sin y)}{du}$

$$1=\cos y\frac{dy}{du}\qquad\text{from (11)}$$

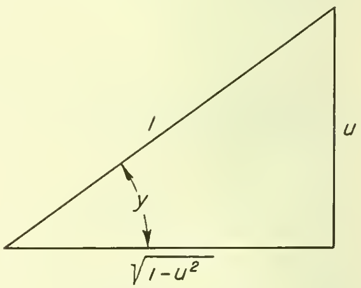

Figure 6–18.—$y=\text{arc}\sin u$.

D. Solve for $\frac{dy}{du}$;

$\frac{dy}{du}=\frac{1}{\cos y}$, but from figure 6–18 we see that $\cos y=\sqrt{1-u^2}$

$$\frac{dy}{du}=\frac{1}{\sqrt{1-u^2}}$$

E. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ from (8) ch 5 and $\frac{dy}{dx}=\frac{d\,(\sin^{-1}u)}{dx}$

$$=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

The question now arises whether the + or − sign should be chosen before the radical. The choice is apparent if we recall that the first derivative of a function is simply the slope of the function. We look at a plot of the function (fig. 6–8) at the point of interest and note the sign of the slope. If the slope is positive, we choose the sign for the radical that will make the first derivative positive.

18. A. $y=\cos^{-1}u=\text{arc cos } u$ (fig. 6–19)

B. u is the cosine of the angle y.

$u=\cos y$

C. $\frac{d\,(u)}{du}=\frac{d\,(\cos y)}{du}$

$1=-\sin y\frac{dy}{du}$ from (12)

D. Solve for $\frac{dy}{du}$;

$\frac{dy}{du}=-\frac{1}{\sin y}$, but from figure 6–19 we see that $\sin y=\sqrt{1-u^2}$

$$\therefore\frac{dy}{du}=-\frac{1.}{\sqrt{1-u^2}}$$

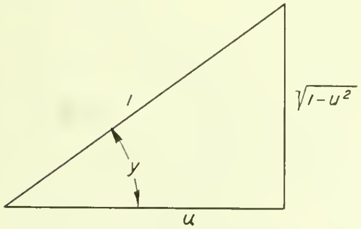

Figure 6–19.—y=arc cosine u.

E. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ from (8) ch 5 and $\frac{dy}{dx}=\frac{d\,(\cos^{-1}u)}{dx}$

$$=-\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

19. A. $y=\tan^{-1}u=\text{arc tan } u$ (fig. 6–20).

B. $\tan y=u$

C. $\frac{d\,(u)}{du}=\frac{d\,(\tan y)}{du}$

$1=\sec^2 y\frac{dy}{du}$ from (13)

D. Solve for $\frac{dy}{du}$;

$\frac{dy}{du}=\frac{1}{\sec^2 y}$,

we see from figure 6–20 that $\sec y=\sqrt{1+u^2}$.

$$\frac{dy}{du}=\frac{1}{1+u^2}$$

E. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ and $\frac{dy}{dx}=\frac{d\,(\tan^{-1}u)}{dx}$

$$=\frac{1}{1+u^2}\frac{du}{dx}$$

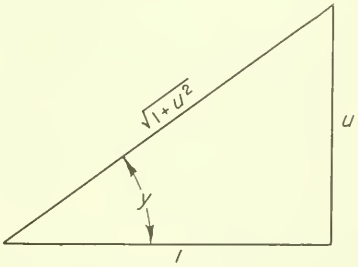

Figure 6–20.—y=arc tan u.

20. A. $y=\cot^{-1}u=\text{arc cot } u$ (fig. 6–21).

B. $u=\cot y$

C. $\frac{d\,(u)}{du}=\frac{d\,(\cot y)}{du}$

$1=-\csc^2 y\frac{dy}{du}$ from (14)

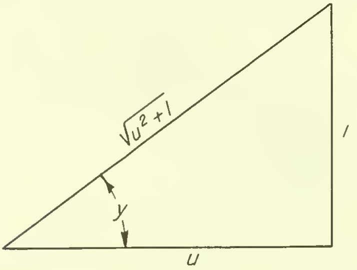

Figure 6-21.—y=arc cot u.

D. Solve for $\frac{dy}{du}$;

$$\frac{dy}{du}=-\frac{1}{\csc^2 y}$$

We see from the figure that $\csc y=\sqrt{u^2+1}$.

$$\therefore \frac{dy}{du}=-\frac{1}{u^2+1}$$

E. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ from (8) ch 5 and $\frac{dy}{dx}=\frac{d\ (\text{arc cot } u)}{dx}$

$$=-\frac{1}{u^2+1}\frac{du}{dx}$$

21. A. $y=\sec^{-1} u=\text{arc sec } u$ (fig. 6–22).

B. $u=\sec y$

C. $\frac{d\ (u)}{du}=\frac{d\ (\sec y)}{du}$

$$1=\sec y \tan y \frac{dy}{du} \qquad \text{from (15)}$$

D. Solve for $\frac{dy}{du}$;

$\frac{dy}{du}=\frac{1}{\sec y \tan y}$, and we know from the figure that $\sec y=u$ and $\tan y=\sqrt{u^2-1}$

$$\therefore \frac{dy}{du}=\frac{1}{u\sqrt{u^2-1}}$$

E. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ from (8) ch 5

$$\therefore \frac{dy}{dx}=\frac{d\ (\text{arc sec } u)}{dx}=\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$$

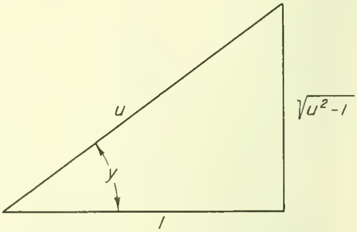

Figure 6-22.—y=arc sec u.

22. A. $y=\csc^{-1} u=\text{arc csc } u$ (fig. 6–23).

B. $u=\csc y$

C. $\frac{d\ (u)}{du}=\frac{d\ (\csc y)}{du}$

$$1=-\csc y \cot y \frac{dy}{du} \qquad \text{from (16)}$$

D. Solve for $\frac{dy}{du}$;

$\frac{dy}{du}=-\frac{1}{\csc y \cot y}$, but from the figure,

$\cot y=\sqrt{u^2-1}$ and $\csc y=u$.

$$\therefore \frac{dy}{du}=-\frac{1}{u\sqrt{u^2-1}}$$

E. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ from (8) ch 5 and $\frac{dy}{dx}=\frac{d\ (\text{arc csc } u)}{dx}$

$$=-\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$$

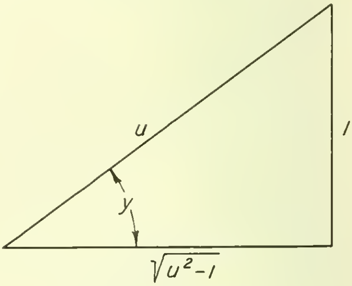

Figure 6-23.—y=arc csc u.

Example 6–4.

Determine the first derivative of the following;

1. $y=\sin^{-1} bx^2$

We know that $\frac{d}{dx}(\sin^{-1} u)=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ from (17)

In this problem $u=bx^2$ and $\frac{du}{dx}=2bx$ from (2) and (4) ch 5

$$\therefore y'=\frac{1}{\sqrt{1-b^2x^4}}2bx=\frac{2bx}{\sqrt{1-b^2x^4}}$$

2. $y=\tan^{-1}\frac{a}{x}$

and $\frac{d(\tan u)}{dx}=\frac{1}{1+u^2}\frac{du}{dx}$ from (19)

$u=\frac{a}{x}$ and $\frac{du}{dx}=\frac{-a}{x^2}$ from (2) and (4) ch 5

$$\therefore \frac{dy}{dx}=\frac{1}{1+\frac{a^2}{x^2}}\left(-\frac{a}{x^2}\right)=\frac{1}{\frac{x^2+a^2}{x^2}}\left(-\frac{a}{x^2}\right)$$

$$\text{or } \frac{dy}{dx}=-\frac{x^2}{x^2+a^2}\left(\frac{a}{x^2}\right)=\frac{a}{x^2+a^2}$$

3. $y=\text{arc sec } 3\pi x$

$$\frac{d}{dx}(\text{arc sec } u)=\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx} \text{ from (21)}$$

$$u=3\pi x \text{ and } \frac{du}{dx}=3\pi \text{ from (4) ch 5}$$

$$\therefore y'=\frac{1}{3\pi x\sqrt{9\pi^2x^2-1}}3\pi=\frac{1}{x\sqrt{9\pi^2x^2-1}}$$

6–11. Differentiation of the Logarithms

23. A. $y=\log_a u$

B. $y+\Delta y=\log_a (u+\Delta u)$

C. $\Delta y=\log_a (u+\Delta u)-\log_a u=\log_a\left(\frac{u+\Delta u}{u}\right)$

$\Delta y=\log_a\left(1+\frac{\Delta u}{u}\right)$

D. $\frac{\Delta y}{\Delta u}=\frac{1}{\Delta u}\log_a\left(1+\frac{\Delta u}{u}\right)$

E. Multiply the right hand side by $\frac{u}{u}$:

$$\frac{\Delta y}{\Delta u}=\frac{u}{\Delta u u}\log_a\left(1+\frac{\Delta u}{u}\right)=\frac{1}{u}\log_a\left(1+\frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}}$$

Before we continue with this derivation, it is important to recall how we defined the number e (paragraph 6–4). We selected the following limit as our definition:

$$e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

Suppose now we let $n=\frac{1}{x}\cdot$ We see that as $x\to\infty$, $n\to 0$. Therefore, an equivalent definition for e would be:

$$e=\lim_{n\to 0}(1+n)^{\frac{1}{n}}.$$

Now we return to part E of our derivation and we see by letting $\frac{\Delta u}{u}=n$, we have $\left(1+\frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}}=(1+n)^{\frac{1}{n}}$.

Further, as $\Delta u\to 0$ so will $\frac{\Delta u}{u}$ which is equal to n.

$$\text{Therefore, } \lim_{\Delta u\to 0}\left(1+\frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}}=\lim_{n\to 0}(1+n)^{\frac{1}{n}}=e.$$

In order to get this limit form we multiplied the right-hand side of part E by $\frac{u}{u}\cdot$

F. $\frac{dy}{du}=\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}=\lim_{\Delta u\to 0}\frac{1}{u}\log_a\left(1+\frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}}$

$$\therefore \frac{dy}{du}=\frac{1}{u}\log_a e$$

G. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ and $\frac{dy}{dx}=\frac{d}{dx}(\log_a u)$

$$=\frac{\log_a e}{u}\cdot\frac{du}{dx}$$

24. A. $y=\log_e u$

From the previous derivation we have $y=\log_a u$ and $\frac{dy}{du}=\frac{d}{du}(\log_a u)=\frac{\log_a e}{u}$ (part F), but now we have $y=\log_e u$

Note the base is e instead of a.

$$\therefore \frac{d}{du}(\log_e u)=\frac{\log_e e}{u}, \text{ but } \log_e e=1$$

so that $\frac{d}{du}(\log_e u)=\frac{1}{u}\cdot$

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B. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ and $\frac{dy}{dx}=\frac{d\,(\log_e u)}{dx}=\frac{1}{u}\frac{du}{dx}$

Example 6–5.

Find the first derivative of each of the following:

1. $y=\log_{10} x^2$

Substituting in (23); $a=10$, $u=x^2$ and $\frac{du}{dx}=2x$

$$\frac{dy}{dx}=\frac{d(\log_{10}x^2)}{dx}=\frac{\log_{10} e}{x^2}\,2x=\frac{2\log_{10} e}{x}$$

2. $y=\log_e x^2$

Substituting in (24); $u=x^2$ and $\frac{du}{dx}=2x$

$$\frac{dy}{dx}=\frac{d\,(\log_e x^2)}{dx}=\frac{1}{x^2}(2x)=\frac{2}{x}$$

3. $y=ln\left(\frac{1-x}{1+x}\right)$

Substituting in (24) $u=\frac{1-x}{1+x}$

$$\frac{dy}{dx}=\frac{d\left(ln\,\frac{1-x}{1+x}\right)}{dx}$$

$$=\frac{1}{\frac{1-x}{1+x}}\cdot\frac{d\left(\frac{1-x}{1+x}\right)}{dx} \qquad (6\text{–}5)$$

Apply (6) ch 5 to complete the process $\frac{d\left(\frac{1-x}{1+x}\right)}{dx}$ by substituting $1-x$ for u and $1+x$ for v in the formula $\frac{d\left(\frac{u}{v}\right)}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

$$\frac{d\left(\frac{1-x}{1+x}\right)}{dx}=\frac{(1+x)\frac{d(1-x)}{dx}-(1-x)\frac{d(1+x)}{dx}}{(1+x)^2}$$

$$=\frac{(1+x)(-1)-(1-x)}{(1+x)^2}$$

$$=\frac{-1-x-1+x}{(1+x)^2}$$

$$=\frac{-2}{(1+x)^2} \qquad (6\text{–}6)$$

Substituting (6–6) in (6–5)

$$\frac{dy}{dx}=\frac{1+x}{1-x}\cdot\frac{-2}{(1+x)^2}=\frac{-2}{(1-x)(1+x)}=\frac{-2}{1-x^2}$$

6–12. Differentiation of Exponential Functions

25. A. $y=a^u$

B. Take the natural logarithm of both sides of the equation in A.

$$ln\ y=u\ ln\ a$$

C. Calculate $\frac{dy}{du}$:

$\frac{1}{y}\frac{dy}{du}=\frac{du}{du}\,ln\ a=ln\ a$ from (24) and (10)

$\frac{dy}{du}=y\ ln\ a$, but $y=a^u$

$\therefore \frac{dy}{du}=a^u\ lna$

D. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ and $\frac{dy}{dx}=\frac{d\,(a^u)}{dx}=a^u\ ln\ a\frac{du}{dx}$

26. A. $y=e^u$

Take the natural logarithm of both sides of the equation A.

B. $ln\ y=u\ ln\ e=u,(ln\ e=1)$

Calculate $\frac{dy}{du}$:

C. $\frac{1}{y}\frac{dy}{du}=\frac{du}{du}=1$ from (24) and (10)

$\therefore \frac{dy}{du}=y=e^u$

D. $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ and $\frac{dy}{dx}=\frac{d\,(e^u)}{dx}=e^u\cdot\frac{du}{dx}$

Example 6–6.

Calculate the first derivatives of the following functions:

1. $y=a^{\ln x}$

Using 25, with $u=ln\ x$ and $\frac{du}{dx}=\frac{1}{x}$,

$$\frac{dy}{dx}=a^{ln\ x}\ ln\ a\frac{1}{x}=\frac{a^{ln\ x}ln\ a}{x}$$

2. $y=e^{\tan x^2}$

$$\frac{dy}{dx}=\frac{d(e^{\tan x^2})}{dx} \qquad (6\text{–}7)$$

$$\frac{d(e^u)}{dx}=e^u\frac{du}{dx} \qquad (26)$$

Substitute $\tan x^2$ from (6–7) for u in (26)

$$\frac{dy}{dx}=\frac{d(e^{\tan x^2})}{dx}=e^{\tan x^2}\cdot\frac{d(\tan x^2)}{dx} \qquad (6\text{–}8)$$

$$\frac{d(\tan u)}{dx}=\sec^2 u\,\frac{du}{dx} \qquad (13)$$

To solve $\frac{d(\tan x^2)}{dx}$ from (6–8) substitute x^2 for u in (13)

$$\frac{d(\tan x^2)}{dx}=\sec^2 x^2\,\frac{d(x^2)}{dx}$$

$$=\sec^2 x^2\, 2x$$

$$=2x\sec^2 x^2 \qquad (6\text{–}9)$$

Substitute (6–9) in (6–8)

$$\frac{dy}{dx}=\frac{d(e^{\tan x^2})}{dx}=e^{\tan x^2}\cdot 2x\sec^2 x^2$$

$$=2x(\sec^2 x^2)e^{\tan x^2}$$

Exercise 6–1.

Differentiate the following transcendental functions.

1. $\sin^3 2x$
2. $e^{ln x}$
3. $e^x \sin^{-1}x$
4. $10^{\tan x}$
5. $\sin^6 2x\cos^5 2x$
6. $t^e+e^{2t}+t^3+\sin t$
7. $ln\sec 3x$
8. $\arctan\frac{\sqrt{x}}{4}$
9. $\text{arc cot}\,\frac{x}{3}$
10. t^3e^{2t}

6–13. Differentials of the Transcendental Functions

We recall that if $y=f(x)$ then the differential of y is given by

$$dy=f'(x)dx.$$

The differentials of the transcendental functions covered in this chapter appear below without further explanation. See section 5–20, chapter 5.

8. $d(\sin u)=\cos u\,du$
9. $d(\cos u)=-\sin u\,du$
10. $d(\tan u)=\sec^2 u\,du$
11. $d(\cot u)=-\csc^2 u\,du$
12. $d(\sec u)=\sec u\tan u\,du$
13. $d(\csc u)=-\csc u\cot u\,du$
14. $d(\sin^{-1}u)=\frac{1}{\sqrt{1-u^2}}du$
15. $d(\cos^{-1}u)=-\frac{1}{\sqrt{1-u^2}}du$
16. $d(\tan^{-1}u)=\frac{1}{1+u^2}du$
17. $d(\cot^{-1}u)=-\frac{1}{1+u^2}du$
18. $d(\sec^{-1}u)=\frac{1}{u\sqrt{u^2-1}}du$
19. $d(\csc^{-1}u)=-\frac{1}{u\sqrt{u^2-1}}du$
20. $d(\log_a u)=\frac{\log_a e}{u}du$
21. $d(ln\,u)=\frac{du}{u}$
22. $d(a^u)=a^u ln\,a\,du$
23. $d(e^u)=e^u du$

Example 6–7.

Calculate cos 46° using differentials. Take cos 45° equal to 0.7071 (see fig. 6–24).

$$y=\cos\theta$$

$dy=-\sin\theta\,d\theta$ from (9) and $\sin\theta=\sin 45°=0.7071$;

$$d\theta=\frac{\pi}{180°}=.0175 \text{ radian or } 1°.$$

$$\therefore dy=(-0.7071)(0.0175)=-0.0124$$

$$y+dy=\cos 45°-0.0124$$

$$y+dy=\cos 46°=0.7071-0.0124=0.6947$$

This result agrees with four place trigonometric tables.

6–14. Converting from Rectangular to Polar Coordinates

We have specified points in a plane by measuring the distances from the axes usually labeled x and y. In figure 6–25 we can specify the exact location of point P by assigning values to the distances labeled x and y. We could also specify the location of the point, P, by assigning a length to the radius, r, and a

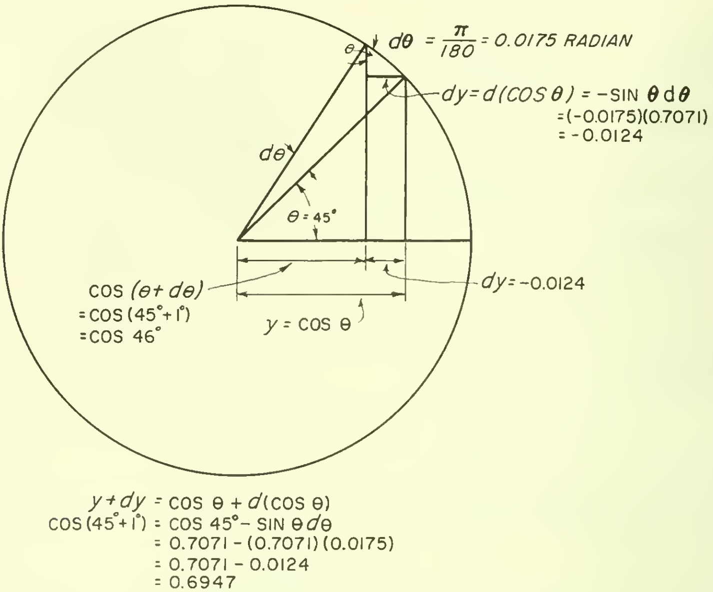

Figure 6-24.—Calculating cos 46° using differentials.

value to the angle, θ. When we use r and θ as our coordinates, we call them the POLAR COORDINATES.

By examining figure 6–25 the equations for converting from one coordinate system to the other should be clear:

$$x=r\cos\theta,\ y=r\sin\theta,\ r^2=x^2+y^2,\ \tan\theta=\frac{y}{x}$$

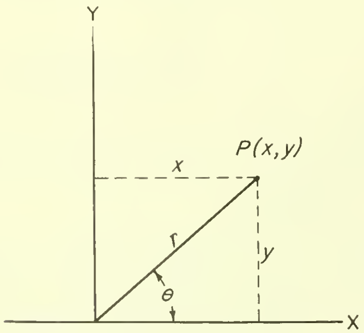

Figure 6–25.—Converting from rectangular to polar coordinates.

Polar coordinates are used when the mathematical operations are simplified by their use. These conditions are usually apparent from the geometry involved.

Example 6–8.

Given $r=f(\theta)$. What is the general expression for the slope of this function at any point?

A. From our previous work we know the slope of any function is $\frac{dy}{dx}$. This implies the use of rectangular coordinates but our given function is in polar coordinates. We must utilize the conversion equations to calculate $\frac{dy}{dx}$.

B. $x=r\cos\theta$

Since r is a function of θ let $u=r, v=\cos\theta$ and $dx=d\theta$ in (5) ch 5 $\left(\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}\right)$

$$\therefore \frac{dx}{d\theta}=\frac{dr}{d\theta}\cos\theta+r(-\sin\theta)$$

C. $y=r\sin\theta$

Let $u=r$, $v=\sin\theta$ and $dx=d\theta$ in (5)

$$\therefore \frac{dy}{d\theta}=\frac{dr}{d\theta}\sin\theta+r\cos\theta$$

D. $$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\frac{dr}{d\theta}\sin\theta+r\cos\theta}{\frac{dr}{d\theta}\cos\theta-r\sin\theta}$$

Let $r'=\frac{dr}{d\theta}$.

$$\frac{dy}{dx}=\frac{r'\sin\theta+r\cos\theta}{r'\cos\theta-r\sin\theta}$$

E. Divide by $r'\cos\theta$;

$$\frac{dy}{dx}=\frac{\tan\theta+\frac{r}{r'}}{1-\frac{r}{r'}\tan\theta}$$

This equation allows us to calculate the slope of $r=f(\theta)$ for all θ's and r's for which the slope exists.

6–15. Circular Motion

The equation of a circle in polar coordinates is simply $r=$constant for the radius is the same everywhere on the circumference of the circle. The distance of a point moving in circular motion, therefore, will remain fixed with respect to the axes about which it is rotating. We can describe the velocity and acceleration associated with this moving point by again utilizing the conversion formulas given in the previous section.

The velocity v_x along the x axis is given by $\frac{dx}{dt}$ and $x=r\cos\theta$. This is similar to (5) in ch 5.

Let $u=r$, $v=\cos\theta$ and $dx=dt$ in $\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$

$$v_x=\frac{dx}{dt}=\frac{dr}{dt}\cos\theta+r(-\sin\theta)\frac{d\theta}{dt}$$

But since r is constant, $\frac{dr}{dt}=0$.

$$\therefore v_x=-r\sin\theta\frac{d\theta}{dt}$$

The velocity along the y axis is given by $\frac{dy}{dt}$ and $y=r\sin\theta$.

Let $u=r$, $v=\sin\theta$ and $dx=dt$ in (5) ch 5.

$$v_y=\frac{dy}{dt}=\frac{dr}{dt}\sin\theta+r\cos\theta\frac{d\theta}{dt}$$

and again $\frac{dr}{dt}=0$.

$$\therefore v_y=r\cos\theta\frac{d\theta}{dt}$$

We measure θ in radians and $\frac{d\theta}{dt}$ will then be in radians/sec. We assign this quantity the Greek letter ω and call it the angular velocity.

$$v_x=-r\omega\sin\theta \text{ and } v_y=r\omega\cos\theta$$

Recalling that $v=\sqrt{v_x^2+v_y^2}$ we have

$$v=\sqrt{r^2\omega^2\sin^2\theta+r^2\omega^2\cos^2\theta} \quad \text{or}$$

$$v=\sqrt{r^2\omega^2(\sin^2\theta+\cos^2\theta)}.$$

$$\sin^2+\cos^2\theta=1$$

$\therefore v=\sqrt{r^2\omega^2}=r\omega=r\frac{d\theta}{dt}$ and is termed the linear velocity.

Strictly speaking, this equation gives the speed, as velocity is a vector and must have a direction. The velocity has a magnitude given by $r\omega$ and is directed along a tangent to the circle at the point (r, θ) in question (fig. 6–26).

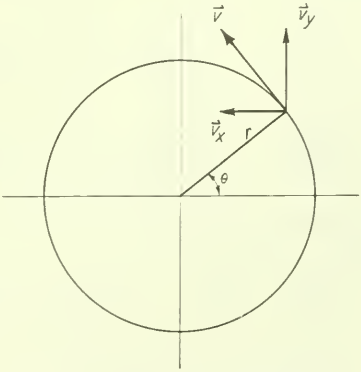

Figure 6–26.—Velocity in circular motion.

We may arrive at expressions for the acceleration in circular motion by differentiating our expressions for v_x and v_y.

$$v_x = -r\frac{d\theta}{dt}\sin\theta \text{ and } \frac{dr}{dt}=0.$$

Let $\left(-r\frac{d\theta}{dt}\right)=u$, $\sin\theta=v$ and $dt=dx$ in (5) ch 5 $\left(\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}\right)$.

$$a_x=\frac{dv_x}{dt}=-r\frac{d^2\theta}{dt^2}\sin\theta-r\frac{d\theta}{dt}\cos\theta\frac{d\theta}{dt}$$

$$a_x=-r\frac{d^2\theta}{dt^2}\sin\theta-r\cos\theta\left(\frac{d\theta}{dt}\right)^2$$

$$v_y=r\frac{d\theta}{dt}\cos\theta \text{ and } \frac{dr}{dt}=0.$$

Let $\left(r\frac{d\theta}{dt}\right)=u$, $\cos\theta=v$, and $dt=dx$ in (5) ch 5 $\left(\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}\right)$

$$a_y=\frac{dv_y}{dt}=r\frac{d^2\theta}{dt^2}\cos\theta-r\frac{d\theta}{dt}\sin\theta\frac{d\theta}{dt}$$

$$a_y=r\frac{d^2\theta}{dt^2}\cos\theta-r\sin\theta\left(\frac{d\theta}{dt}\right)^2$$

$$a=\sqrt{a_x^2+a_y^2}$$

Substituting the expressions for a_x and a_y and simplifying yields;

$$a_x^2=\left[-r\frac{d^2\theta}{dt^2}\sin\theta-r\cos\theta\left(\frac{d\theta}{dt}\right)^2\right]^2$$

$$=r^2\left(\frac{d^2\theta}{dt^2}\right)^2\sin^2\theta+2r^2\frac{d^2\theta}{dt^2}\left(\frac{d\theta}{dt}\right)^2\sin\theta\cos\theta+r^2\cos^2\theta\left(\frac{d\theta}{dt}\right)^4$$

$$a_y^2=\left[r\frac{d^2\theta}{dt^2}\cos\theta-r\sin\theta\left(\frac{d\theta}{dt}\right)^2\right]^2$$

$$=r^2\left(\frac{d^2\theta}{dt^2}\right)^2\cos^2\theta-2r^2\left(\frac{d^2\theta}{dt^2}\right)\left(\frac{d\theta}{dt}\right)^2\sin\theta\cos\theta+r^2\sin^2\theta\left(\frac{d\theta}{dt}\right)^4$$

$$a=\Bigg[\underbrace{r^2\left(\frac{d^2\theta}{dt^2}\right)^2\sin^2\theta}_{(1)}+\underbrace{2r^2\frac{d^2\theta}{dt^2}\left(\frac{d\theta}{dt}\right)^2\sin\theta\cos\theta}_{(2)}$$

$$+\underbrace{r^2\cos^2\theta\left(\frac{d\theta}{dt}\right)^4}_{(3)}+\underbrace{r^2\left(\frac{d^2\theta}{dt^2}\right)^2\cos^2\theta}_{(4)}-\underbrace{2r^2\left(\frac{d^2\theta}{dt^2}\right)\left(\frac{d\theta}{dt}\right)^2\sin\theta\cos\theta}_{(5)}$$

$$+\underbrace{r^2\sin^2\theta\left(\frac{d\theta}{dt}\right)^4}_{(6)}\Bigg]^{1/2}$$

terms (2) and (5) cancel out; factor $(\sin^2\theta+\cos^2\theta)$ from terms (1) and (4) and terms (3) and (6) and substitute $(\sin^2\theta+\cos^2\theta)=1$

$$a=\sqrt{r^2\left(\frac{d^2\theta}{dt^2}\right)^2(\sin^2\theta+\cos^2\theta)+r^2\left(\frac{d\theta}{dt}\right)^4(\sin^2\theta+\cos^2\theta)}$$

$a=\sqrt{r^2\left(\frac{d^2\theta}{dt^2}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^4}$ which is the expression for the total acceleration.

More often the components are utilized as the total acceleration expression is too cumbersome. These components are no longer a_x and a_y, however, as these were lost in the derivation process. We now have the tangential (a_t) and the radial (a_r) components of the acceleration corresponding to $r\frac{d^2\theta}{dt^2}$ and $r\left(\frac{d\theta}{dt}\right)^2$ respectively.

$$a_t^2=r^2\left(\frac{d^2\theta}{dt^2}\right)^2 \text{ so that}$$

$a_t=\sqrt{a_t^2}=\sqrt{r^2\left(\frac{d^2\theta}{dt^2}\right)^2}=r\frac{d^2\theta}{dt^2}=r\frac{d\omega}{dt}$ and is the tangential acceleration.

$$a_r^2=r^2\left(\frac{d\theta}{dt}\right)^4 \text{ so that}$$

$a_r=\sqrt{a_r^2}=\sqrt{r^2\left(\frac{d\theta}{dt}\right)^4}=r\left(\frac{d\theta}{dt}\right)^2=r\omega^2$ and is the radial acceleration.

In the expression for a_t the $\frac{d\omega}{dt}$ is generally assigned the Greek letter α and is called the angular acceleration.

$$a_t=r\alpha \qquad a_r=r\omega^2$$

The equivalent expressions are:

$$a_t=r\frac{d^2\theta}{dt^2}=r\frac{d\omega}{dt}=r\alpha \text{ and}$$

$$a_r=r\left(\frac{d\theta}{dt}\right)^2=\frac{v^2}{r}=r\omega^2.$$

The tangential acceleration vector is directed along a tangent to the circle and the radial acceleration vector is directed along a radius of the circles pointing from the circumference towards the center of the circle (fig. 6–27).

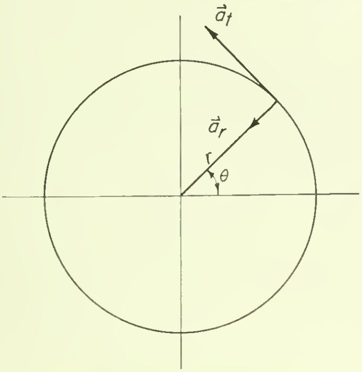

Figure 6–27.—Acceleration in circular motion.

Example 6–9.

A particle moves in a circle according to

$$\theta = 3t^2 - t + 1$$

where θ is measured in radians and t in seconds. If the circle is 8 feet in diameter, calculate the magnitude of the angular velocity (ω) linear velocity $(r\omega)$, tangential acceleration $(r\alpha)$, and radial acceleration $(r\omega^2)$ at $t=2$ sec.

A. The angular velocity is $\omega = \frac{d\theta}{dt}$.

$$\therefore \frac{d\theta}{dt} = \frac{d(3t^2 - t + 1)}{dt} = \omega = 6t - 1 \ \frac{\text{radians}}{\text{seconds}}$$

and if $t=2$ sec

$$\omega = \frac{d\theta}{dt} = 11 \frac{\text{radians}}{\text{seconds}}$$

B. The linear velocity is given by $v = r\omega$ and at $t=2$ sec, $\omega = 11$ rad/sec.

$$\therefore v = (4 \text{ feet})(11 \text{ rad/sec}) = 44 \text{ feet/sec.}$$

C. The tangential acceleration is $a_t = r\alpha$.

$$\alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{d(6t+1)}{dt} = 6 \text{ rad/sec}^2$$

and is independent of the time.

$$\therefore a_t = r\alpha = (4 \text{ feet})(6 \text{ rad/sec}^2) = 24 \text{ ft/sec}^2$$

D. The radial acceleration is $a_r = r\omega^2$ and at $t=2$, $\omega = 11$ rad/sec.

$$\therefore a_r = (4 \text{ feet})\ (11 \text{ rad/sec})^2 = 484 \text{ feet/sec}^2.$$

Example 6–10.

(See figure 6–28.) Wheel A is turning according to $\theta = t^3 - 1$ with θ in radians and t in seconds. Wheel A causes wheel B to move without slipping. How much greater is the magnitude of the linear velocity of point P_2 with respect to that of point P_1 when $t = 1$ second?

A. We calculate the angular velocity of wheel A.

$$\omega_A = \frac{d\theta}{dt} = \frac{d(t^3-1)}{dt} 3t^2 \text{ and } t = 1 \text{ sec.}$$

$$\therefore \omega_A = 3 \text{ rad/sec.}$$

B. We may now calculate the linear velocity of P_1 located a distance 1.5 feet from the center of wheel A.

$$v_1 = r_1\ \omega_A = (1.5 \text{ feet})\ (3 \text{ rad/sec}) = 4.5 \text{ ft/sec}$$

C. Now it is necessary to determine the linear velocity of point C which is the point of contact between the two wheels. Since there is no slipping between the wheels, the linear velocity of a point on the circumference of both wheels is the same.

$$v_C = r_2\omega_A = (4 \text{ ft})\ (3 \text{ rad/sec}) = 12 \text{ ft/sec}$$

D. As v_C is common to both wheels, we may now obtain ω_B.

$$v_C = r_C\omega_B$$
$$\omega_B = \frac{v_C}{r_C}$$

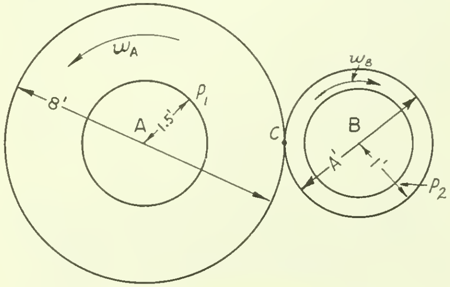

Figure 6–28.—Example 6–9.

And on wheel B, $r_c = 2$ ft.

$$\omega_B = \frac{12 \text{ ft/sec}}{2 \text{ ft}} = 6 \text{ rad/sec.}$$

Note another way to calculate ω_B is to use the ratio of the diameters involved. This is valid when the circumferences are in contact and there is no slipping as in the usual case of two gears.

$\frac{\omega_A}{\omega_B} = \frac{D_B}{D_A}$ and $\omega_A = 3$ rad/sec, $D_B = 4$ ft, and $D_A = 8$ ft.

$$\omega_B = \frac{D_A \omega_A}{D_B} = \frac{(8)(3)}{4} = 6 \text{ rad/sec.}$$

E. The linear velocity of point P_2 is now obtainable. Note that P_2 is located 1 foot from the center of wheel B.

$$v_2 = r_2\,\omega_B = (1 \text{ ft})(6 \text{ rad/sec}) = 6 \text{ ft/sec}$$

Comparing v_2 and v_1 we see that v_2 is $6-4.5 = 1.5$ ft/sec greater than v_1 at $t = 1$ second.

Example 6–11.

What is the algebraic expression for the angular velocity (ω) of a point that moves in such a manner that $\theta = e^{2t} \tan 3t$ describes its angular displacement for all t?

$$\omega = \frac{d\theta}{dt} = \frac{d(e^{2t} \tan 3t)}{dt} \tan 3t \frac{d}{dt}(e^{2t}) + e^{2t} \frac{d}{dt}(\tan 3t)$$

$$\omega = \tan 3t(e^{2t})(2) + e^{2t}(\sec^2 3t)(3)$$

$$\omega = e^{2t}(2 \tan 3t + 3 \sec^2 3t)$$

6–16. Simple Harmonic Motion

Figure 6–29 shows point P moving about point O in a circle of radius r. It moves with a constant angular velocity ω with P′ indicating the projection of P on the x axis. The angular displacement measured counterclockwise from the x axis is indicated by θ and is equal to ωt.

We will not concern ourselves with the projection of P on the y axis, but a similar argument applies as that for the projection on the x axis.

Notice as P moves about its circular path, the projection P′ moves back and forth on the x axis so that the line OP′ varies in length between zero and r. This length is given for any angle by;

$$x = r \cos \theta = r \cos \omega t$$

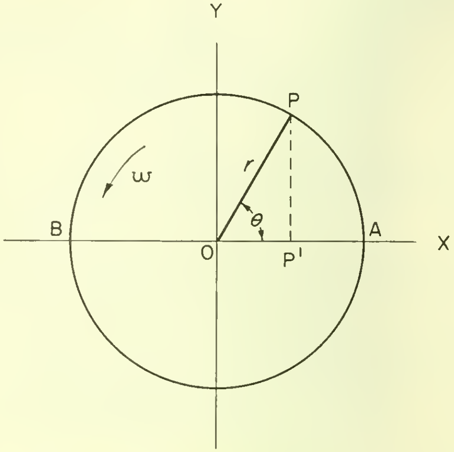

Figure 6–29.—The movement of P about O.

The linear velocity of P′ is $\frac{dx}{dt}$ or;

$$v = \frac{dx}{dt} = \frac{d(r \cos \omega t)}{dt} = -r\omega \sin \omega t \text{ from (12)}$$

The maximum value of the velocity, disregarding the sign, is $r\omega$ which occurs when $\theta = 90°$ or $270°$; that is, when P is directly above or below the center O.

The linear acceleration of P′ is $\frac{d^2x}{dt^2}$ or

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{d(-r\omega \sin \omega t)}{dt} = -r\omega^2 \cos \omega t$$

and since

$$x = r \cos \omega t,$$

$$a = \omega^2 x$$

This equation indicates that the acceleration is proportional to the displacement. The proportionality constant is $-\omega^2$ with the negative sign indicating the acceleration is directed oppositely to the displacement.

From the equation $a = -r\omega^2 \cos \omega t$, we see the maximum value for the acceleration is $r\omega^2$ disregarding the sign.

$$a_{max} = r\omega^2$$

This maximum value occurs at $\theta=0°$ and 180° or when P is at A or B.

The amount of time that elapses as P moves through one complete circle is called the period. In this time P′ will go from A to B and back to A. We have 2π radians in one circle or 360° of angular displacement, and the angular velocity is $\omega \frac{\text{radians}}{\text{seconds}}$.

$$\text{Period}=T=\frac{2\pi}{\omega} \text{ sec per cycle.}$$

The frequency with which P passes through point A is given by the reciprocal of the period:

$$f=\frac{1}{T}=\frac{\omega}{2\pi} \text{ cycles/sec.}$$

In this argument we began at zero time ($t=0$) with P at point A so that $x=r \cos \omega t$ for any t. Suppose now we begin with P angularly offset from the x axis at $t=0$ in this case, the projection of P the x axis at $t=0$ by some angle say α (fig. 6–30). At $t=0$ in this case, the projection of P on the x axis defines a line $OP'=r \cos \alpha$. At any time later the projection is given by:

$$x=r \cos (\omega t+\alpha)$$

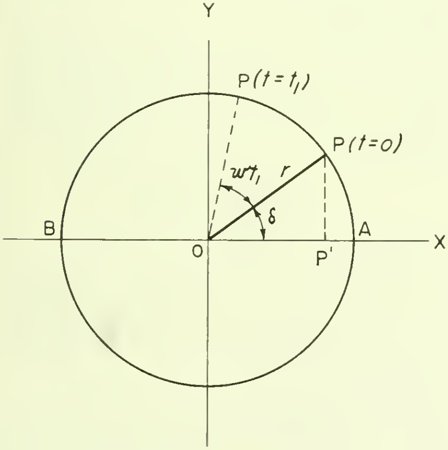

Figure 6–30.—Projection of P.

and in particular at $t=t_1$,

$$x=r \cos (\omega t_1+\alpha).$$

The initial angular offset (α) is called the phase angle.

Example 6–12.

A body is undergoing simple harmonic motion with the displacement given by

$$x=4 \cos (\pi t+\frac{\pi}{2}) \text{ feet.}$$

What is the velocity (V), acceleration (A), and displacement (x) at $t=\frac{1}{4}$ sec? Determine the phase angle (α), amplitude (r) of motion, angular velocity, (ω) and period (T).

A. We first determine the displacement at $t=\frac{1}{4}$ sec.

$$x=4 \cos \left(\frac{\pi}{4}+\frac{\pi}{2}\right)=4 \cos \left(\frac{3\pi}{4}\right)$$

$$\frac{3\pi}{4} \text{ radians}=135° \qquad x=4 \cos 135°$$

$\cos 135°=-\frac{\sqrt{2}}{2}$ so that the displacement $x=-(4)\left(\frac{\sqrt{2}}{2}\right)=-2\sqrt{2}$ feet. The negative sign indicates displacement to the left of the origin.

B. The velocity at $t=\frac{1}{4}$ sec is found from $\frac{dx}{dt}$.

$$v=\frac{dx}{dt}=-4\pi \sin \left(\pi t+\frac{\pi}{2}\right)$$

$$v=-4\pi \sin \left(\frac{3\pi}{4}\right)=-4\pi \sin 135°$$

$$\sin 135°=\frac{\sqrt{2}}{2} \text{ so that } v=-2\pi \sqrt{2} \text{ ft/sec.}$$

The negative sign indicates the velocity is directed to the left.

C. Find the acceleration at $t=\frac{1}{4}$ second from $\frac{d^2x}{dt^2}$.

$$a=\frac{d^2x}{dt^2}=-4\pi^2\cos\left(\pi t+\frac{\pi}{2}\right)=-4\pi^2\cos 135°$$

$$a=(-4\pi^2)\left(-\frac{\sqrt{2}}{2}\right)=2\pi^2\sqrt{2}\text{ ft/sec}^2$$

Hence the acceleration is positive and directed to the right.

D. The general equation for the displacement with a phase angle is given by

$$x=r\cos(\omega t+\alpha).$$

Comparing $x=4\cos\left(\pi t+\frac{\pi}{2}\right)$ with the general form allows us to determine the remaining desired quantities.

Phase angle $=\alpha=\frac{\pi}{2}$ radians

Angular velocity $=\omega=\pi$ radians/sec
Amplitude $=r=4$ ft
Period $=\text{T}=\frac{2\pi}{\omega}=\frac{2\pi}{\pi}=2$ seconds

Example 6–13.

The electric field at a particular point in space reaches its maximum value of 100 volts every microsecond. What is the rate of change of the electric field at this location 2.5×10^{-7} seconds after its maximum value is reached assuming the field varies according to simple harmonic motion.

A. We are told that E_{max} is reached every 10^{-6} second. Hence the period (T) is 10^{-6} second.

$$\therefore\omega=\frac{2\pi}{\text{T}}=2\pi\times10^6\text{ rad/sec}$$

B. At the beginning of the cycle $e=E_{max}$ so that the equation must be of the form

$$e=E_m\cos\omega t.$$

Substituting $t=0$ does yield $e=E_{max}$ since cos $(0)=1$.

C. We determine the rate of change of e from $\frac{de}{dt}$.

$$\frac{de}{dt}=\frac{d(E_m\cos\omega t)}{dt}=E_m\frac{d(\cos\omega t)}{dt}\qquad\begin{matrix}\text{from (4) ch 5}\\\text{from (12)}\end{matrix}$$

$$=-\omega E_m\sin\omega t\text{ and}$$

$$\omega=2\pi\times10^6\text{ rad/sec}$$

$$E_m=100\text{ volts}$$

$$t=2.5\times10^{-7}\text{ seconds}$$

$$\frac{de}{dt}=(-2\pi\times10^6)(100)\sin[(2\pi\times10^6)(2.5\times10^{-7})]$$

$$\frac{de}{dt}=-2\pi\times10^8\sin(5\pi\times10^{-1})$$

$$\sin(5\pi\times10^{-1})=\sin\frac{\pi}{2}=\sin 90°=1$$

$$\frac{de}{dt}=-2\pi\times10^8\text{ volts/sec}$$

The electric field is DECREASING at the rate of $2\pi\times10^8$ volts/sec.

6–17. L'Hospital's Rule

In section 5–7 (chapter 5) we discussed limits of indeterminant forms. We are now prepared to deal with the indeterminant forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Let us call the numerator $g(x)$, and the denominator $h(x)$. Then we have $\frac{g(x)}{h(x)}$ as the form for which we wish to determine the limit.

We state L'Hospital's Rule without proof and in general terms.

If $\lim\limits_{x\to a}\frac{g(x)}{h(x)}$ is of the indeterminant forms $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$, differentiate the numerator and denominator separately and evaluate this new limit which will be equal to the limit of the original expression. In equation form:

$$\lim_{x\to a}\frac{g(x)}{h(x)}=\lim_{x\to a}\frac{g'(x)}{h'(x)}.$$

If $\lim\limits_{x\to a}\frac{g'(x)}{h'(x)}$ is again of the indeterminant forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, reapply the rule. The rule holds for finite and infinite values of a.

Example 6–14.

Determine $\lim\limits_{x\to0}\frac{\sin x}{x}$. We recall in section 6–8 we used geometric constructions and trigonometric relationships to calculate this limit. We found it necessary to go through these steps in order to derive the formula for the derivative of sin x. Now that we developed this formula, a direct application of L'Hospital's Rule gives $\lim_{x\to 0}\dfrac{\sin x}{x}$

$$=\lim_{x\to 0}\frac{\dfrac{d(\sin x)}{dx}}{\dfrac{dx}{dx}}=\lim_{x\to 0}\frac{\cos x}{1}=1.$$

Of course this tool was not available to us in section 6–8.

Example 6–15.

Find $\lim_{x\to 0}\dfrac{e^x+e^{-x}-2}{x-\tan x}$.

This is of the form $\dfrac{0}{0}$.

$$\therefore \lim_{x\to 0}\frac{e^x+e^{-x}-2}{x-\tan x}=\lim_{x\to 0}\frac{\dfrac{d(e^x+e^{-x}-2)}{dx}}{\dfrac{d(x-\tan x)}{dx}}$$

$$=\lim_{x\to 0}\frac{e^x-e^{-x}}{1-\sec^2 x}=\frac{0}{0}$$

Apply the rule once more; from (10), (13), and (26)

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{1-\sec^2 x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{-2\sec^2 x\tan x}=\frac{1+1}{0}=\infty$$

from (1), (2), (7) and (15)

The limit is ∞ and the original expression becomes unbounded as $x\to 0$.

Example 6–16.

Find $\lim_{x\to\infty}\dfrac{2x^2}{e^x}$. This type is $\frac{\infty}{\infty}$.

$$\therefore \lim_{x\to\infty}\frac{2x^2}{e^x}=\lim_{x\to\infty}\frac{\dfrac{d(2x^2)}{dx}}{\dfrac{d(e^x)}{dx}}=\lim_{x\to\infty}\frac{4x}{e^x}=\frac{\infty}{\infty} \text{ so that}$$

$$\lim_{x\to\infty}\frac{4x}{e^x}=\lim_{x\to\infty}\frac{4}{e^x}=0.$$

Example 6–17.

Find $\lim_{x\to\infty} x^{-2}\ln x$. This may not appear to be of the form $\dfrac{0}{0}$ or $\frac{\infty}{\infty}$ for it takes the form $0-\infty$. But rearranging gives;

$$\lim_{x\to\infty}\frac{\ln x}{\dfrac{1}{x^2}} \text{ or the form } \frac{\infty}{\infty}.$$

$$\therefore \lim_{x\to\infty}\frac{\ln x}{\dfrac{1}{x^2}}=\lim_{x\to\infty}\frac{\dfrac{1}{x}}{-2x^{-3}}=\lim_{x\to\infty}\frac{1}{-2x^{-2}}=\frac{x^2}{-2}=\infty$$

Care must be taken to differentiate the numerator separately from the denominator and not as a quotient when applying L'Hospital's Rule.

6–18. Examples of Application

Example 6–18.

In a certain circuit the voltage across a $30\mu f$ capacitor is found to vary as $v=20e^{-2t}\sin 60t$. What is the current flowing in the capacitor at any time t?

A. We calculate the expression for the current flowing in a capacitor from

$$i=C\frac{dv}{dt}$$

$$\therefore\ i=(30\times 10^{-6})\ \frac{d(20e^{-2t}\sin 60t)}{dt}$$

$$\frac{dv}{dt}=\frac{d(20e^{-2t}\sin 60x)}{dt}$$

from (5), (11) and (26)

In (5) ch 5 $\dfrac{d(uv)}{dx}=v\dfrac{du}{dx}+u\dfrac{dv}{dx}$

let $u=20e^{-2t}$, $v=\sin 60t$ and $dx=dt$

$$\frac{dv}{dt}=\sin 60t\ \frac{d(20e^{-2t})}{dt}+20e^{-2t}\ \frac{d(\sin 60t)}{dt} \qquad (6\text{–}10)$$

$$\frac{d(20e^{-2t})}{dt}=20\ \frac{d(e^{-2t})}{dt}=20e^{-2t}(-2) \qquad (6\text{–}11)$$

$$\frac{d(\sin 60t)}{dt}=\cos 60t\ \frac{d(60t)}{dt}=60\cos 60t \qquad (6\text{–}12)$$

Substituting (6–11) and (6–12) in (6–10)

$$\frac{dv}{dt}=(\sin 60t)20e^{-2t}(-2)+20e^{-2t}60\cos 60t$$

$$=20(-2\sin 60te^{-2t}+60\cos 60te^{-2t})$$

$$i=c\frac{dv}{dt}$$

$$=30\times 10^{-6}\times 20(-2\sin 60te^{-2t}+60\cos 60te^{-2t})$$

$$=30\times 10^{-6}\times 20\times 2 3e^{-2t}(30\cos 60t-\sin 60t)$$

$$=1.2\times 10^{-3}e^{-2t}(30\cos 60t-\sin 60t)$$

Example 6–19.

In an RL series circuit the current is given by $i=\frac{V}{R}(1-e^{\frac{-Rt}{L}})$. What is the voltage across the inductor at time $t=\frac{L}{R}$? The source voltage is V, inductance is L, and R is the resistance.

A. The voltage at any time across an inductor is $V=L\frac{di}{dt}$.

B. $\frac{di}{dt}=\left(-\frac{V}{R}\right)\left(-\frac{R}{L}\right)e^{\frac{-Rt}{1}}=\frac{V}{L}e^{\frac{-Rt}{L}}$ from (26)

and if $t=\frac{L}{R}$ we have

$$\frac{di}{dt}=\frac{V}{L}e^{-\frac{RL}{RL}}=\frac{V}{eL}=0.37\frac{V}{L}$$

C. $V=L\frac{di}{dt}=\frac{(L)(0.37\,V)}{L}=0.37\,V$ at time $t=\frac{L}{R}$.

This time is called the time constant in an RL circuit.

Example 6–20.

In an electron tube the number of electrons evaporated per unit area of emitting (cathode) surface is related to the absolute temperature of the emitting material and a quantity b that is a measure of the work an electron must perform in leaving the surface according to the equation

$$i=AT^2e^{-\frac{b}{T}}\text{ amperes.}$$

A and b are constants related to the material of the cathode and T is the absolute temperature of the cathode. Determine the rate of change of the current with respect to the absolute temperature.

A. $\frac{di}{dT}=AT^2\left(\frac{b}{T^2}\right)e^{-\frac{b}{T}}+e^{-\frac{b}{T}}2AT$

$\frac{di}{dT}=Ae^{-\frac{b}{T}}(2T-b)$ amperes per degree absolute

Example 6–21.

The inductance of the toroid shown in figure 6–31 may be calculated from

$$L=\frac{\mu_oN^2h}{2\pi}\,ln\,\frac{r_o}{r_i}.$$

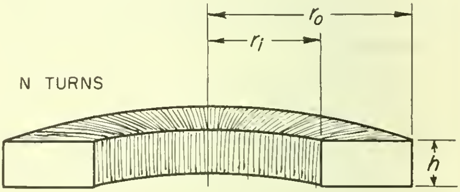

Figure 6–31.—A toroid.

The quantities μ_o and 2π are constants and all other quantities are as indicated in the figure.

Determine the rate of change of inductance as we vary the outside radius and hold the inside radius, number of turns, and height constant.

A. We wish to determine $\frac{dL}{dr_0}$.

$$\frac{dL}{dr_0}=\frac{\mu_0N^2h}{2\pi}\left(\frac{1}{\frac{r_0}{r_i}}\right)\left(\frac{1}{r_i}\right).$$ We recall

$$\frac{d\,(ln\;u)}{dx}=\frac{1}{u}\frac{du}{dx}.$$

$$\frac{dL}{dr_0}=\frac{\mu_0N^2h}{2\pi}\left(\frac{r_i}{r_0}\right)\left(\frac{1}{r_i}\right)=\frac{\mu_0N^2h}{2\pi r_0}$$

The rate of change of the inductance increases as r_0 decreases.

CHAPTER 7

INTEGRATION

7–1. Introduction

Chapters 5 and 6 have introduced us to inverse functions. In example 5–19 chapter 5 we learned that with every function there is a related inverse function such as the inverse trigonometric functions or the inverse exponential functions; the latter are the logarithms.

Just as there are inverse functions, there are also inverse operations. For example, in mathematics, subtraction is the inverse of addition, and division is the inverse of multiplication. This chapter and chapter 8 are concerned with the inverse process of differentiation; INTEGRATION.

The student will learn the rules for integrating the standard forms; they are analogous to the standard forms of differentiation. Further, he will learn techniques to transform nonstandard forms to standard forms which he may then integrate directly. Along with the pure mechanical manipulations, he will investigate numerous applications in an attempt to clarify his theory and justify his labor.

Integration is a more difficult process to grasp than is differentiation. The rules are not as foolproof nor as exact. More effort will be required on the part of the student, but patience will be rewarded by satisfying results.

7–2. Definition of the Integral

Integration is defined as the inverse of differentiation. In other words, we are given the derivative of some function and our task is to determine what function we differentiated to acquire the given derivative. Herein lies the difficulty, for our ability to "guess" this function is dependent upon our understanding of the differentiation process. For this reason a review of the standard forms of differentiation should be undertaken at this point.

7–3. The Symbol of the Integral

The symbol for the integral is an elongated S, $\int$. Perhaps this was thought appropriate as integration can be shown to be the limit of a sum. Hence, the elongated S comes from the first letter of the word sum.

7–4. An Interpretation of the Integral

An integral may represent many things. It may be a physical process or a pure mathematical process. It may be real or abstract. It may represent a surface area of some figure, or a plane area, a volume, or even something being "summed" in ten-dimensional space. Our interpretation will be limited however, to more earthly quantities. Let us investigate the plane area as one interpretation of the integral.

Current is defined as the net rate at which electrons pass a given cross section of a conductor.

In equation form

$$i=\frac{dq}{dt}.$$

Suppose we are given a situation in which the current varies uniformly from 0 to 8 amperes in 5 seconds (fig. 7–1). We wish to determine the

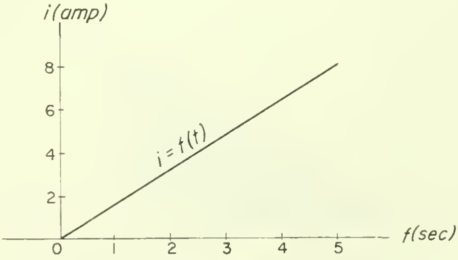

Figure 7–1.—$i=f(t)$.

quantity of charge that passes a given cross section of the conductor in the 5-second period.

We multiply both sides of the previous equation by the differential dt.

$$i\,dt = \frac{dq}{dt}\,dt \text{ or } dq = i\,dt,$$

and using increments rather than differentials we get $\Delta q \approx i\,\Delta t$. We want to determine Δq and will say that $dq \approx \Delta q$ so that $\Delta q = i\,\Delta t$ to a close enough approximation.

Looking at the figure we see the curve describing $i = f(t)$ is a straight line, so we use an average value of the current to calculate Δq.

$$\therefore \Delta q = \frac{i_f - i_i}{2}\,\Delta t = \left(\frac{8}{2}\right)(5)$$

$$\Delta q = 20 \text{ amp} \cdot \text{sec} = 20 \text{ coulombs.}$$

Now we will calculate the plane area shown shaded in figure 7–2. The area is a triangle and the formula for the area of a triangle is

$$A = \frac{1}{2}bh.$$

The base, b, is 5 sec. and the height, h, is 8 amperes.

$$\therefore A = \frac{1}{2}(5)(8) = 20 \text{ amp} \cdot \text{sec.}$$

and

$$A = 20 \text{ coulombs.}$$

The area of the shaded portion beneath the curve is equal to the charge flowing past a given cross section of conductor in 5 seconds.

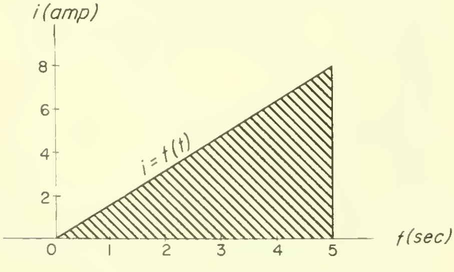

Figure 7–2.—Area under $i = f(t)$.

We have actually summed all the little areas representing dq (fig. 7–3) to finally arrive at the total charge, $q = 20$ coulombs. Notice that the area of the small element labeled dq, is a rectangle whose area is given by

$$dq = i\,dt$$

where i in this case represents the average current during the differential time interval, dt. If we sum all these elements over any interval of time, we get

$$q = \int i\,dt.$$

That is, the total charge is equal to the integral of $i\,dt$.

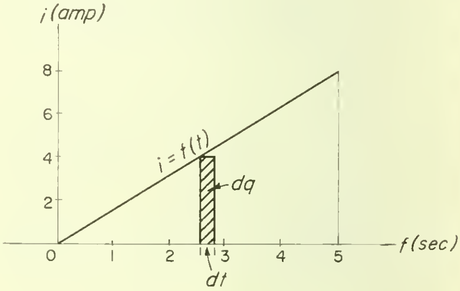

Figure 7–3.—Shaded area dq.

A more detailed discussion of "the area under the curve interpretation" of the integral will appear in chapter eight.

7–5. The Constant of Integration

$$\text{If } y = x^2, \frac{dy}{dx} = f'(x) = 2x$$

and $dy = 2x dx$. $(dy = f'(x)\,dx)$. Then $\int 2x dx = x^2$.

Similarly, $y = x^3$ and $dy = 3x^2 dx$. Then

$$\int 3x^2 dx = x^3.$$

We may generalize this to $\int f'(x) dx = f(x)$ where $f'(x)$ is called the INTEGRAND. This generalized form must be true since, as we have said, integration is the inverse of the differentiation process.

In integration we work with differentials rather than derivatives so we may restate integration to be the process of finding the function from the differential of the function.

The relation

$$\int f'(x)dx = f(x),$$

indicates a method for verifying each integration. If we take the differential of the function resulting from the integration, we must obtain the function which we integrated.

$$df(x) = f'(x)dx$$

$$\int 2xdx = x^2 \text{ and } d(x^2) = 2xdx$$

$$\int 3x^2dx = x^3 \text{ and } d(x^3) = 3x^2dx$$

Suppose we perform the integration indicated by $\int 2xdx$. We obtain x^2 as before. However, $d(x^2) = 2xdx$, $d(x^2 + \sqrt{2}) = 2xdx$, $d(x^2 - 6) = 2xdx$, and in fact $d(x^2 + C) = 2xdx$ where C is any constant, positive or negative.

We ask then, does $\int 2xdx = x^2$, $x^2 + \sqrt{2}$, $x^2 - 6$, or $x^2 + C$? Since we may take the differential of ANY of these functions and arrive at $2xdx$, we must conclude that they are ALL correct. Therefore, we choose the most general result, $\int 2xdx = x^2 + C$ which includes all constants.

A statement that follows directly from this result is that all functions that differ by any constant have the same derivative (differential).

We generalize the previous results:

$$\int f'(x)dx = f(x) + C.$$

C is called, the CONSTANT OF INTEGRATION. It arises from the fact that any function of the form $f(x) + C$ has the differential $f'(x)dx$. The constant of integration may be determined from conditions specified in each particular problem.

The quantity, $f(x) + C$, is called the indefinite integral. The name suggests that no particular value for the integral may be assigned until C is determined and a value is assigned to $f(x)$. The indefinite integral, even with C determined, is a function of some variable and thus remains indefinite.

7–6. Evaluation of the Constant of Integration

The techniques used to evaluate the constant of integration can best be illustrated by working the following examples.

Example 7–1.

Find the equation of the curve whose first derivative is two times the independent variable.

A. Let x be the independent variable.

$$\therefore \frac{dy}{dx} = f'(x) = 2x \text{ so that } dy = f'(x)dx = 2xdx$$

B. We obtain the desired result by integrating the expression for dy.

$$y = \int dy = \int f'(x)dx = \int 2xdx; \; y = x^2 + C$$

C. $d(x^2 + C) = 2xdx$ verifying the result of the integration. We have not obtained a particular solution; we have obtained only a general solution as a different curve results for each value of C. (See figure 7–4.)

D. If we further specify that $x = 0$, $y = 6$, we may obtain a definite value for C and hence a particular solution.

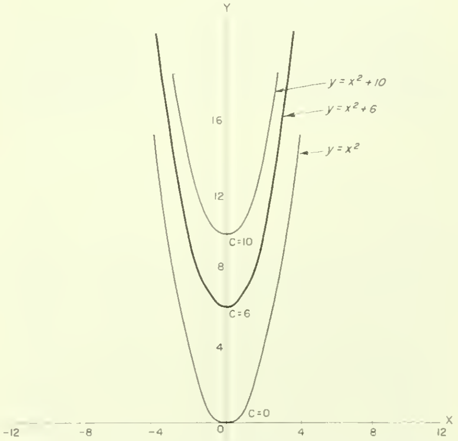

Figure 7–4.—The family of curves plotted from $y = x^2 + C$.

$y=x^2+C$ and when $x=0$, $y=6$.

$\therefore 6=0+C$ and $C=6$.

The complete solution is the curve described by $y=x^2+6$.

This example provides us with a geometrical interpretation for the constant of integration. The general result of the integration yielded x^2+C, the indefinite integral. When plotted, the result was a family of curves each differing from another curve by a constant. Thus, each has the SAME derivative which was one of our given premises.

We were given additional information, called INITIAL or BOUNDARY CONDITIONS, from which we were able to find the particular curve from the family of curves. We may also describe this procedure as determining the particular solution from the general solution.

Example 7–2.

Find the equation for the quantity of charge delivered to the plate circuit of an electron tube in which the equation of the plate current is $i=3t^2$. At $t=0$ the charge on the plate due to a plate capacitor is 2×10^{-6} coulomb.

A. $i=\frac{dq}{dt}\frac{\text{coulombs}}{\text{sec}}$

We form the expression for the differential of charge, dq. $dq=\frac{dq}{dt}dt=idt$

The variable of integration is t, indicated by dt. This requires that we obtain i as a function of t before we may integrate. We are given this information, $i=3t^2$

$\therefore q=\int dq=\int idt=\int 3t^2dt$ and $q=t^3+C$ which is the general solution.

B. We have the initial condition from which we may calculate the constant of integration. When $t=0$, $C=2$ microcoulombs (2×10^{-6} coulombs).

$\therefore$ 2 microcoulombs $=0+C$ and $C=2$ microcoulombs which is the desired result.

The constant of integration in this example represents the initial charge retained due to a plate capacitor. This provides us with a PHYSICAL INTERPRETATION, for the constant of integration since charge is a physical quantity.

7–7. The Definite Integral

So far we have dealt solely with the indefinite integrals. We know that the indefinite integral has the general form $\int f'(x)dx=f(x)+C$.

There are two identifying characteristics;

(1) A constant of integration is required to be added with each integration and

(2) the result of the integration is a function of a variable and has no definite value until a value is assigned this variable. This assumes the constant of integration has been determined.

We now investigate the form and properties of the DEFINITE INTEGRAL. As the name implies, it has a definite value AND THUS IS A NUMBER INDEPENDENT OF ANY VARIABLE.

The definite integral $\int_a^b f'(x)dx$, is read, "The integral from a to b of $f'(x)dx$."

The letter b is called the upper limit and a, the lower limit of the integral. These limits have nothing to do with the limits of a function as described in chapter five. Rather, they are the limits of the region over which we are integrating $f'(x)dx$. As indicated earlier, this region may be a line segment, an area, a volume, time, or any other quantity.

The integral is evaluated in the following manner;

$$\int_a^b f'(x)dx=f(b)+C-[f(a)+C]=f(b)-f(a)$$

Notice that the constant of integration is eliminated when evaluating a definite integral. We see that the result is a number completely independent of any variable. The number is the difference between the two numbers $f(b)$ and $f(a)$. We subtract the function evaluated at the lower limit from the function evaluated at the upper limit.

We may illustrate the definite integral is independent of any variable with an example.

Example 7–3.

Evaluate the definite integral $\int_{x=1}^{x=2} 2xdx$ and show that it is the same as the integral $\int_{t=1}^{t=2} 2tdt$ showing that the definite integral is independent of x and t.

$$\int_{x=1}^{x=2} 2x\,dx = x^2\Big|_1^2 = 4-1 = 3 \text{ and}$$

$$\int_{t=1}^{t=2} 2t\,dt = t^2\Big|_1^2 = 4-1 = 3.$$

Therefore, this definite integral equals the NUMBER 3.

We normally omit the notation for the variable on the integral symbol and simply write

$$\int_1^2 2x\,dx = x^2\Big|_1^2 = 4-1 = 3.$$

However, the notation may be essential to clearly indicate which variable we mean if more than one is included under the integral sign.

Example 7–4.

Evaluate the integral $q=\int_{t=1}^{t=3} i\,dt$ if $i=3t^2$.

We include the notation for the variable in this case to indicate that we are interested in the region involving t and not i.

$$\therefore q = \int_{t=1}^{t=3} i\,dt = \int_1^3 3t^2\,dt$$

$$q = t^3\Big|_1^3 = 27-1 = 26$$

Two general properties of the definite integral should be understood.

(1) Interchanging the limits of an integral changes the sign in front of the integral.

$$\int_a^b f'(x)dx = -\int_b^a f'(x)dx$$

Example 7–5.

Illustrate the first general property of the definite integral.

$$\int_1^2 2x\,dx = -\int_2^1 2x\,dx$$

$$x^2\Big|_1^2 = -x^2\Big|_2^1$$

$$4-1 = -1+4$$

$$3 = 3$$

(2) The integral over any region may be divided into the sum of any number of integrals each covering a portion of the entire region.

$$\int_a^b f'(x)dx = \int_a^c f'(x)dx + \int_c^b f'(x)dx$$

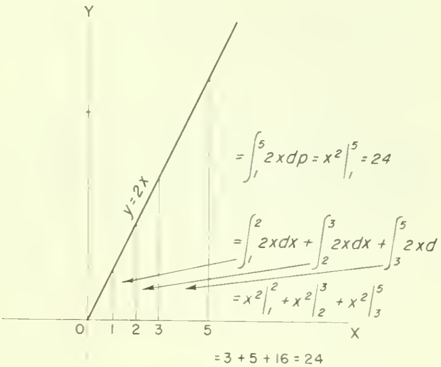

Figure 7–5.—Example 7–6.

Example 7–6.

Illustrate the second general property of the definite integral (fig. 7–5).

$$\int_1^5 2x\,dx = \int_1^2 2x\,dx + \int_2^3 2x\,dx + \int_3^5 2x\,dx$$

$$x^2\Big|_1^5 = x^2\Big|_1^2 + x^2\Big|_2^3 + x^2\Big|_3^5$$

$$25-1 = (4-1)+(9-4)+(25-9)$$

$$24 = 3+5+16$$

$$24 = 24$$

7–8. Rules for Integrating Standard Forms

We recall that the verification of integration lies in differentiating the result of integration to obtain the integrand. Therefore, $\int f'(x)dx = f(x)$ and verifying we have

$$\frac{d}{dx}[f(x)] = f'(x) \text{ or using}$$

differentials, $df(x) = f'(x)dx$.

The derivations of the rules for integrating standard forms consist of finding the function whose differential is one of the standard forms.

The integrals for a number of the standard forms are listed here for easy reference. Derivations are given along with examples of a few of the forms. These rules must be memorized.

The Standard Forms

The letters u, v, and w are functions of x and C is a constant

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1. $\int du = u + C$

The integral of a sum (difference) is equal to the sum (difference) of the integrals.

2. $\int (du + dv + dw) = \int du + \int dv + \int dw = u + v + w + C$

The integral of the product of a constant and a variable is equal to the product of the constant and the integral of the variable. That is, a constant may be moved across the integral sign without changing the value of the problem.
problem.

3. $\int a du = a \int du = au + C$

4. $\displaystyle\int u^n du = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1$

5. $\displaystyle\int \frac{du}{u} = ln\ u + C$

6. $\displaystyle\int a^u du = \frac{a^u}{ln\ a} + C$

7. $\int e^u du = e^u + C$

8. $\int \sin u du = -\cos u + C$

9. $\int \cos u\ du = \sin u + C$

10. $\int \sec^2 u\ du = \tan u + C$

11. $\int \csc^2 u\ du = -\cot u + C$

12. $\int \sec u \tan u\ du = \sec u + C$

13. $\int \csc u \cot u\ du = -\csc u + C$

14. $\int \tan u\ du = -ln \cos u + C = ln \sec u + C$

15. $\int \cot u\ du = ln \sin u + C$
$= -ln \csc u + C$

16. $\int \sec u\ du = ln (\sec u + \tan u) + C$

17. $\int \csc u\ du = -ln (\csc u + \cot u) + C$

18. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$

19. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$

20. $\displaystyle\int \frac{du}{au + b} = \frac{1}{a}\ ln\ (au + b) + C$

21. $\displaystyle\int \frac{u\ du}{au^2 + b} = \frac{1}{2a}\ ln\ (au^2 + b) + C$

22. $\displaystyle\int \frac{du}{au^2 + b} = \frac{1}{\sqrt{ab}} \tan^{-1} \left(u\sqrt{\frac{a}{b}}\right) + C$

23. $\displaystyle\int (au^2 + b)^n u du = \frac{1}{2a} \frac{(au^2 + b)^{n+1}}{n+1} + C;\ n \neq -1$

24. $\displaystyle\int \sin a u du = -\frac{1}{a} \cos au + C$

25. $\displaystyle\int (au + b)^n du = \frac{1}{a(n+1)} + C$

26. $\displaystyle\int \cos a\ u\ du = \frac{1}{a} \sin au + C$

7–9 Derivations of the Rules (1–4)

1. $\displaystyle\int du = u + C$

Proof: $d(u + C) = du$

$$\therefore \int du = u + C$$

2. $\displaystyle\int (du + dv + dw) = \int du + \int dv + \int dw$

Proof: $d(u + v + w + C) = du + dv + dw$

$$\therefore \int (du + dv + dw) = u + v + w + C$$

From rule 1 we see that

$$\int du + \int dv + \int dw = (u + C_1) + (v + C_2) + (w + C_3)$$

Let $C = C_1 + C_2 + C_3$

$$\therefore \int du + \int dv + \int dw = u + v + w + C$$

We have then

$$\int (du + dv + dw) = \int du + \int dv + \int dw = u + v + w + C$$

3. $\displaystyle\int a du = a \int du = au + C$

Proof: $d(au + C) = ad(u + C_1) = adu\ (aC_1 = C)$

$$\therefore \int adu = a \int du = au + C$$

4. $\displaystyle\int u^n du = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1$

Proof: $\displaystyle d\left(\frac{u^n + 1}{n+1} + C\right) = \frac{(n+1)u^n}{n+1} du = u^n du$

$$\therefore \int u^n du = \frac{u^{n+1}}{n+1} + C$$

We may not use this rule for $n = -1$ as the denominator, $n + 1$, would be zero and division by zero is undefined

Example 7–6.

Evaluate the following integrals utilizing rules 1 through 4.

1. $\int x^2dx=\dfrac{x^{2+1}}{2+1}+C=\dfrac{x^3}{3}+C$ (Rule 4)

2. $\int 3dx=3x+C$ (Rules 1 and 3)

3. $\int 2x^{1/2}dx=\dfrac{2x^{1/2+1}}{\frac{1}{2}+1}+C=\dfrac{2x^{3/2}}{\frac{3}{2}}$

$$+C=\frac{4x^{3/2}}{3}+C \text{ (Rules 3 and 4)}$$

4. $\int \dfrac{3}{x^2}\,dx=3\int x^{-2}dx=\dfrac{3x^{-2+1}}{-2+1}+C$

$$=\frac{3x^{-1}}{-1}+C=-\frac{3}{x}+C \text{ (Rules 3 and 4)}$$

5. $\int (x^{5/2}+x^{1/4})dx=\dfrac{x^{7/2}}{\frac{7}{2}}+\dfrac{x^{5/4}}{\frac{5}{4}}+C$

$$=\frac{2x^{7/2}}{7}+\frac{4x^{5/4}}{5}+C \text{ (Rules 2 and 4)}$$

6. $\int (3x^2+3x)^{1/2}(6x+3)dx$

At first sight example (6) looks unlike any forms covered to this point. However, suppose we let

$$u=3x^2+3x \text{ and } du=(6x+3)dx.$$

We then have the form $\int u^n du$ with $n=\dfrac{1}{2}$. Therefore, using rule 4 we integrate.

$$\int (3x^2+3x)^{1/2}(6x+3)dx=\frac{(3x^2+3x)^{1/2+1}}{\frac{1}{2}+1}+C$$

$$=\frac{2(3x^2+3x)^{3/2}}{3}+C$$

7. $\int (3x^2+3x)^{1/2}(2x+1)dx$

Again we let $u=3x^2+3x$ so that $du=(6x+3)dx$. However, we don't have du in this case, but if we multiply $(2x+1)dx$ by 3 we get $(6x+3)dx$ which is du.

Rule 3 states we may move a constant back and forth across an integral sign without altering the evaluation. Therefore, let us multiply our integral by $\dfrac{3}{3}=1$.

$$\frac{3}{3}\int (3x^2+3x)^{1/2}(2x+1)dx$$

We choose to take the 3 in the numerator inside the integral sign and multiply it by $(2x+1)$ and leave the 3 in the denominator outside of the integral. By rule 3, we have not altered the problem.

$$\frac{1}{3}\int (3x^2+3x)^{1/2}(6x+3)dx$$

We now have the form $a\int u\,du$ with $a=\dfrac{1}{3}$, $u=3x^2+3x$, and $du=(6x+3)dx$. We may integrate directly using rule 4.

$$\frac{1}{3}\int (3x^2+3x)^{1/2}(6x+3)dx=\frac{1}{3}\frac{(3x^2+3x)^{3/2}}{\frac{3}{2}}$$

$$+C=\frac{2(3x^2+3x)^{3/2}}{9}+C$$

8. $\dfrac{xdx}{(x^2+7)^2}$

Let $u=x^2+7$ and then $du=2xdx$. The numerator would be du if we multiplied it by 2. If we multiply by 2, we must also divide by 2 so as not to change the value. We now have the form

$a\int u^n du$ with $a=\dfrac{1}{2}$, $u=x^2+7$, $du=2xdx$, and $n=-2$

$$\therefore \frac{1}{2}\int\frac{2xdx}{(x^2+7)^2}=\frac{1}{2}\int (x^2+7)^{-2}\,2xdx=\frac{1}{2}\frac{(x^2+7)^{-1}}{-1}$$

$$+C=\frac{-1}{2(x^2+7)}+C$$

We may prove the answers to the previous eight problems by differentiating. In so doing we must obtain the integrand of the original integral. For example in problem 1,

$$\frac{d}{dx}\left(\frac{x^3}{3}+C\right)=x^2 \text{ which}$$

is the integrand. The reader should prove the seven remaining problems in a like manner. It would be well to remember that any integration may be verified by differentiation.

Exercise 7–1.

Perform the following integrations and prove the results by differentiation.

1. $\int (ax+b)dx$
2. $\int (x+x^2+x^3)dx$
3. $\int 2x^{1/3}dx$
4. $\int \frac{4dx}{\sqrt{2x+1}}$
5. $\int \left(x^4+\frac{x}{2}\right)^3 \left(2x^3+\frac{1}{4}\right) dx$

7–10. Derivation of Rules (5–13)

We continue the derivation of the integration rules with rule 5.

5. $\int \frac{du}{u} = ln\ u + C$

Proof: $d(ln\ u+C) = \frac{1}{u}\ du$

$$\therefore \int \frac{du}{u} = ln\ u + C$$

6. $\int a^u du = \frac{a^u}{ln\ a} + C$

Proof: $d(a^u+C) = a^u\ ln\ a\ du$

so that $\int a^u ln\ a\ du = a^u + C_1$

But since $ln\ a$ is a constant, we have $ln\ a \int a^u du = a^u + C_1$

Transposing, $\int a^u du = \frac{a^u}{ln\ a} + \frac{C_1}{ln\ a}$

Let $\frac{C_1}{ln\ a} = C$

$$\therefore \int a^u du = \frac{a^u}{ln\ a} + C$$

7. $\int e^u du = e^u + C$

Proof: $d(e^u + C) = e^u du$

$$\therefore \int e^u du = e^u + C$$

8. $\int \sin u\ du = -\cos u + C$

Proof: $d(\text{Cos}\ u + C) = -\sin u\ du$
so that $d(-\text{Cos}\ u + C) = +\sin u\ du$

$$\therefore \int \sin u\ du = -\cos u + C$$

9. $\int \cos u\ du = \sin u + C$

Proof: $d(\sin u + C) = \cos u\ du$

$$\therefore \int \cos u\ du = \sin u + C$$

10. $\int \sec^2 u\ du = \tan u + C$

Proof: $d(\tan u + C) = \sec^2 u\ du$

$$\therefore \int \sec^2 u\ du = \tan u + C$$

11. $\int \csc^2 u\ du = -\cot u + C$

Proof: $d(\cot u + C) = -\sec^2 u\ du$
so that $d(-\cot u + C) = \csc^2 u\ du$

$$\therefore \int \text{Csc}^2\ u\ du = -\cot u + C$$

12. $\int \sec u \tan u\ du = \sec u + C$

Proof: $d(\sec u + C) = \sec u \tan u\ du$

$$\therefore \int \sec u \tan u\ du = \sec u + C$$

13. $\int \text{Csc}\ u \cot u\ du = -\csc u + C$

Proof: $d(\csc u + C) = -\csc u \cot u\ du$

so that $d(-\csc u + C) = \csc u \cot u\ du$

$$\therefore \int \csc u \cot u\ du = -\csc u + C$$

Example 7–7.

Evaluate the following integrals.

1. $\int \sin 2x\ dx$

We have $u=2x$ so that $du=2dx$. We multiply the integral by $\frac{2}{2}$ to acquire the form $\int \sin u\, du$.

$$\frac{1}{2}\int (\sin 2x)2dx=\frac{1}{2}(-\cos 2x)+C$$
$$=\frac{1}{2}-\cos 2x+C \quad \text{(Rule 8)}$$

2. $\int e^{x^3}x^2dx$

Let $u=x^3$, $du=3x^2dx$. We seek the form $\int e^u du$ so that we multiply the integral by $\frac{3}{3}$.

$$\frac{1}{3}\int e^{x^3}\,3x^2\,dx=\frac{e^{x^3}}{3}+C \quad \text{(Rule 7)}$$

3. $\int \frac{2dx}{2x+3}$

Let $u=2x+3$, $du=2dx$

$\therefore$ using rule 5

$$\int \frac{2dx}{2x+3}=ln\,(2x+3)+C$$

4. $\int \frac{6x^2-4x+2\,dx}{3x+1}$

This integral presents a new problem for the power of x that appears in the numerator is greater than the power in the denominator. We may generalize the procedure to follow with the statement: Whenever we see an integral in which the variable appears to a higher power in the numerator than in the denominator, we must divide the numerator by the denominator and integrate the result.

$$6x^2-4x+2\div 3x+1=2x-2+\frac{4}{3x+1}$$

$$\int\left(2x-2+\frac{4}{3x+1}\right)dx=\int 2xdx-\int 2dx+\int\frac{4dx}{3x+1}$$
$$=x^2-2x+\frac{4}{3}ln\,(3x+1)+C$$

(Rules 2, 4, and 5)

For

$$\int\frac{4dx}{3x+1}=\frac{4}{3}\int\frac{3dx}{3x+1}=\frac{4}{3}ln\,(3x+1)+C_1$$

(The constant C_1 is incorporated with C.)

5. $\int \cos\frac{2}{5}t\,dt$

$$u=\frac{2}{5}t,\ du=\frac{2dt}{5}$$
$$\therefore \int\cos\frac{2}{5}t\,dt=\frac{5}{2}\int\frac{2}{5}\cos\frac{2}{5}t\,dt$$
$$=\frac{5}{2}\sin\frac{2}{5}t+C$$

6. $\int \sec^2 3x\,dx=\frac{1}{3}\int 3\sec^2 3x\,dx$

$$=\frac{1}{3}\tan 3x+C$$

7. $\int 7xb^{x^2}\,dx$

$$u=x^2,\ du=2x\,dx$$
$$\therefore \frac{7}{2}\int\frac{2}{7}7x\,b^{x^2}\,dx=\frac{7}{2}\int 2xb^{x^2}\,dx$$
$$=\frac{7}{2}\frac{b^{x^2}}{ln\,b}+C \quad \text{(Rule 6)}$$

8. $\int \csc\frac{t}{2}\cot\frac{t}{2}\,dt$

$$u=\frac{t}{2},\ du=\frac{dt}{2}$$
$$\therefore 2\int\frac{1}{2}\left(\csc\frac{t}{2}\cot\frac{t}{2}\right)dt=-2\csc\frac{t}{2}+C$$

9. $\int \cos^3 x\sin x\,dx$

Let $u=\cos x$, $du=-\sin x\,dx$ and $n=3$.

We have the form $\int u^n du$ except for the negative sign which we place as follows:

$$-\int(\cos^3 x)(\sin x\,dx)=-\frac{\cos^4 x}{4}+C$$

10. $\int \tan^2 2x\sec^2 2x\,dx$

$u=\tan 2x$, $du=\sec^2 2x\,d(2x)=(\sec^2 2x)(2\,dx)$ from formulas (13) Ch. 6 and (10) Ch. 5.

$\therefore \frac{1}{2}\int 2\tan^2 2x\sec^2 2x\,dx$ is of the form

$$a\int u^n\,du$$

$$\frac{1}{2}\int 2\tan^2 2x\sec^2 2x\,dx=\frac{\tan^3 2x}{6}+C$$

Exercise 7–2.

Perform the following integrations and verify the results by differentiation.

1. $\int K^{2t}dt$
2. $\int x \cos \frac{x^2}{2}\, dx$
3. $\int \frac{vdv}{av^2+b}$
4. $\int \sin^2 3x \cos 3xdx$
5. $\int \frac{x^2-4x-10}{\frac{x}{2}+1}\, dx$
6. $\int \sec^2 x\, e^{\tan x}\, dx$
7. $\int x \csc^2 (x^2-1)dx$
8. $\int \sec x \tan x\, e^{\sec x}\, dx$

7–11. Derivation of the Rules (14–19)

We conclude the derivations of the rules for integration with rules 14 through 19.

14. $\int \tan udu = -\ln \cos u + C = \ln \sec u + C$

Proof: $\int \tan udu = \int \frac{\sin udu}{\cos u}$

This last integral is almost of the form $\int \frac{dv}{v}$ for if $v = \cos u$, $du = -\sin udu$. We need introduce only the minus sign to obtain $\int \frac{dv}{v}$.

$$\therefore -\int \frac{-\sin udu}{\cos u} = -\ln \cos u + C$$

Another form may be acquired by a slightly different approach.

$$\int \tan udu = \int \frac{\sec u \tan udu}{\sec u} \left(\text{multiply by } \frac{\sec u}{\sec u}\right)$$

This last integral is immediately of the form $\int \frac{dv}{v}$ with $v = \sec u$ and $dv = \sec u \tan udu$.

$$\therefore \int \frac{\sec u \tan udu}{\sec u} = \ln \sec u + C$$

15. $\int \cot udu = \ln \sin u + C = -\ln \csc u + C$

Proof: $\int \cot udu = \int \frac{\cos udu}{\sin u}$ and immediately we have $\int \frac{dv}{v}$.

$$\therefore \int \frac{\cos udu}{\sin u} = \ln \sin u + C$$

We proceed to the second form.

$$\int \cot udu = -\int -\frac{\csc u \cot udu}{\csc u} \text{ and}$$

$$= -\ln \csc u + C$$

16. $\int \sec udu = \ln (\sec u + \tan u) + C$

Proof: Multiply by $\frac{\sec u + \tan u}{\sec u + \tan u}$ so that we get $\int \frac{(\sec^2 u + \sec u \tan u)du}{\sec u + \tan u}$. This integral is of the form $\int \frac{dv}{v}$ for if $v = \sec u + \tan u$, $dv = (\sec u \tan + \sec^2 u)du$

$$\therefore \int \frac{(\sec^2 u + \sec u \tan u)du}{\sec u + \tan u}$$

$$= \ln (\sec u + \tan u) + C$$

17. $\int \csc udu = -\ln (\csc u + \cot u) + C$

Proof: Following similar steps as in the proof of rule 16; multiply $\int \csc udu$ by $\frac{\csc u + \cot u}{\csc u + \cot u}$

$$\int \csc udu = \int \frac{(\csc u + \cot u) \csc udu}{\csc u + \cot u}$$

$$= -\int -\frac{(\csc^2 udu + \cot u \csc udu)}{\csc u + \cot u}$$

The integral is now in the form

$$\int \frac{dv}{v} = \ln v + C$$

where $v = \csc u + \cot u$ and

$$dv = d (\csc u) + d (\cot u)$$
$$= -\csc u \cot udu + (-\csc^2 udu)$$

$$\int \csc udu = -\ln (\csc u + \cot u)$$

18. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}+C$

Proof: We know $d(\sin^{-1} v) = \frac{dv}{\sqrt{1-v^2}}$

Therefore, $d\left(\sin^{-1}\frac{u}{a}+C\right) = \frac{d\left(\frac{u}{a}\right)}{\sqrt{L-\left(\frac{u}{a}\right)^2}} = \frac{\frac{1}{a}du}{\sqrt{\frac{a^2-u^2}{a^2}}}$

$= \frac{a}{a}\frac{du}{\sqrt{a^2-u^2}} = \frac{du}{\sqrt{a^2-u^2}}$

$$\therefore \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}+C$$

19. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}+C$

Proof: We know $d(\tan^{-1} v) = \frac{dv}{1+v^2}$ so that

$$d\left(\frac{1}{a}\tan^{-1}\frac{u}{a}\right) = \frac{1}{a}\frac{d\left(\frac{u}{a}\right)}{1+\left(\frac{u}{a}\right)^2} = \frac{1}{a}\frac{\frac{1}{a}du}{\frac{a^2+u^2}{a^2}} = \frac{du}{a^2+u^2}$$

$$\therefore \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan-1\frac{u}{a}+C$$

Example 7–8.

Carry out the indicated integrations.

1. $\int \frac{dx}{\sqrt{4-x^2}}$

This is of the form $\int \frac{dx}{\sqrt{a^2-u^2}}$ with $a=2$. $u=x$, and $du=dx$.

$$\therefore \int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2}+C$$

2. $\int \tan\frac{x}{9}dx$

$u=\frac{x}{9}$, $du=\frac{dx}{9}$ and we have the form $\tan u du$.

$$\therefore \int \tan\frac{x}{9}dx = 9\int\frac{1}{9}\tan\frac{x}{9}dx$$

$$= -9\ ln\cos\frac{x}{9}+C$$

or

$$= 9\ ln\sec\frac{x}{9}+C$$

3. $\int \csc 2\theta d\theta$

$$u=2\theta,\ d\theta=2d\theta$$

$$\therefore \frac{1}{2}\int 2\csc 2\theta d\theta = -\frac{1}{2}ln(\csc 2\theta+\cot 2\theta)+C$$

4. $\int \frac{2xdx}{9+x^4}$

This is of the form $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}+C$

$$u=x^2,\ du=2xdx,\ \text{and } a=3$$

$$\therefore \int \frac{2xdx}{9+x^4} = \frac{1}{3}arc\tan\frac{x^2}{3}+C$$

Notice that rules 18 and 19 contain two terms in the denominator; rule 18 has the difference between two squares (a^2-u^2) and rule 19 has the sum of two squares (a^2+u^2). There are cases where three terms are contained in the denominator, but may be converted to the sum or difference of two squares by completing the square. (See NavPers 10071 revised.)

5. $\int \frac{dx}{\sqrt{5+4x-x^2}} = \int \frac{dx}{\sqrt{5-x^2+4x+4-4}}$

$$= \int \frac{dx}{\sqrt{9-(x-2)^2}}$$

This is of the form $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}+C$ with $u=x-2$, $du=dx$, and $a=3$.

$$\therefore \int \frac{dx}{\sqrt{9-(x-2)^2}} = \sin^{-1}\frac{x-2}{3}+C$$

6. $\int \frac{dx}{\frac{5}{4}+x^2+x} = \int \frac{dx}{1+\frac{1}{4}+x^2+x} = \int \frac{dx}{1+\left(x+\frac{1}{2}\right)^2}$

This is of the form $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}+C$

Let $u=x+\frac{1}{2}$, $du=dx$, and $a=1$

$$\therefore \int \frac{dx}{1+\left(x+\frac{1}{2}\right)^2} = \tan^{-1}\left(x+\frac{1}{2}\right)+C$$

7–12. Integrating Powers of Trigonometric Functions

Integrating powers of trigonometric functions require special techniques. Let us first investigate the general form $\int \sin^p u \cos^q u\, du$.

Case 1. Either p or q or both are positive odd integers.

Example 7–9.

$$\int \sin^3 u \cos^2 u\, du \qquad p=3, q=2$$

$$\int \sin^3 u \cos^2 u\, du = \int \sin^2 u \sin u \cos^2 u\, du$$

(Recall that $\sin^2 u = 1 - \cos^2 u$)

$$= \int (1-\cos^2 u)\cos^2 u \sin u\, du$$

$$= \int (\cos^2 u - \cos^4 u)\sin u\, du$$

$$\int (\cos^2 u - \cos^4 u)\sin u\, du$$

$$= \int \cos^2 u \sin u\, du - \int \cos^4 u \sin u\, du$$

The last two integrals are of the form $\int u^n du$ except for the $-$ sign for $d(\cos u) = -\sin u\, du$.

$$\therefore -\int \cos^2 u(-\sin u\, du) - (-)\int \cos^4 u(-\sin u\, du)$$

$$= -\frac{\cos^3 u}{3} + \frac{\cos^5 u}{5} + C$$

Example 7–10.

$$\int \sin^2 u \cos^5 u\, du \qquad p=2, q=5$$

$$\int \sin^2 u \cos^5 u\, du = \int \sin^2 u \cos^4 u \cos u\, du$$

$$= \int \sin^2 u\,(1-\sin^2 u)^2 \cos u\, du$$

(Recall $\cos^2 u = 1 - \sin^2 u$)

$$= \int (\sin^2 u - 2\sin^4 u + \sin^6 u)\cos u\, du$$

Each is now of the form $\int u^n du$ for

$$d(\sin u) = \cos u\, du$$

$\therefore$ Integrating directly we get

$$\frac{\sin^3 u}{3} - 2\frac{\sin^5 u}{5} + \frac{\sin^7 u}{7} + C$$

Similar techniques hold if both p and q are positive and even.

Case 2. Both p and q are positive and even integers. We will utilize the following trigonometric identities from chapter 1.

A. $\sin u \cos u = \frac{1}{2}\sin 2u$

B. $\sin^2 u = \frac{1}{2} - \frac{1}{2}\cos 2u$

C. $\cos^2 u = \frac{1}{2} + \frac{1}{2}\cos 2u$

Example 7–11.

$$\int \sin^2 u \cos^2 u\, du$$

Using formulas A and B we obtain

$$\int \sin^2 u \cos^2 u\, du = \int \frac{1}{4}\sin^2 2u\, du = \frac{1}{4}\int \sin^2 2u\, du$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4u\right)du = \frac{1}{8}\int du - \frac{1}{8}\int \cos 4u\, du$$

(multiply the last term by $\frac{4}{4}$ to convert to the standard form $\int \cos u\, du = \sin u$)

$$= \frac{1}{8}\int du - \frac{1}{8}\cdot\frac{1}{4}\int (\cos 4u)(4du)$$

$$= \frac{u}{8} - \frac{1}{32}\sin 4u + C$$

Example 7–12.

$$\int \sin^2 u \cos^4 u\, du$$

Rearranging we acquire

$\int (\sin^2 u \cos^2 u)\cos^2 u\, du$. Using formulas (A) and (C),

$$\frac{1}{4}\int \sin^2 2u\left(\frac{1}{2} + \frac{1}{2}\cos 2u\right)du$$

$$= \frac{1}{8}\int \sin^2 2u\, du + \frac{1}{8}\int \sin^2 2u \cos 2u\, du$$

Apply formula (B) to the first term.

$$=\frac{1}{8}\int\left(\frac{1}{2}-\frac{1}{2}\cos 4u\right)du+\frac{1}{8}\int \sin^2 2u\cos 2u\,du$$

$$=\frac{1}{16}\int du-\frac{1}{16}\cdot\frac{1}{4}\int(\cos 4u)(4du)$$

$$+\frac{1}{8}\cdot\frac{1}{2}\int(\sin^2 2u)(\cos 2u)(2du)$$

$$=\frac{u}{16}-\frac{\sin^3 2u}{64}+\frac{\sin^3 2u}{48}+C$$

We may be confronted with the problem of integrating the products $\sin p\,x\cos q\,x$, $\sin p\,x\sin q\,x$, or $\cos p\,x\cos q\,x$. These forms may be handled by using the following formulas from chapter 1.

D. $\sin px\cos qx=\frac{1}{2}\{\sin[(p+q)x]+\sin[(p-q)x]\}$

E. $\sin px\sin qx=\frac{1}{2}\{\cos[(p-q)x]-\cos[(p+q)x]\}$

F. $\cos px\cos qx=\frac{1}{2}\{\cos[(p+q)x]+\cos[(p-q)x]\}$

Example 7–13.

$\int \sin 3x\cos 2x\,dx$

Using formula D we obtain,

$$\frac{1}{2}\int(\sin 5x+\sin x)dx=\frac{1}{2}\int\sin 5x dx+\frac{1}{2}\int\sin x\,dx$$

$$=-\frac{\cos 5x}{10}-\frac{\cos x}{2}+C$$

Example 7–14.

$\int \cos 5x\cos 3x\,dx$

Using formula F,

$$\frac{1}{2}\int(\cos 8x+\cos 2x)dx=\frac{1}{2}\int\cos 8x\,dx+\frac{1}{2}\int\cos 2x\,dx$$

$$=\frac{\sin 8x}{16}+\frac{\sin 2x}{4}+C$$

We have dealt with only the sine and cosine functions in introducing the techniques used to integrate powers of trigonometric functions and products of trigonometric functions with different angular expressions. The reader is encouraged to consult more detailed texts on the integral calculus to learn the techniques associated with the remainder of the trigonometric functions.

Exercise 7–3.

Evaluate the following integrals. Prove the results correct by differentiation.

1. $\int \sec 2x\,dx$
2. $\int \frac{t\,dt}{\sqrt{\frac{9-t^4}{4}}}$
3. $\int 3\theta\cot 2\theta^2\,d\theta$
4. $\int \sin^3 2x\cos^2 2x\,dx$
5. $\int \frac{dx}{13-4x+x^2}$
6. $\int x^2\tan 3x^3\,dx$
7. $\int \sin\frac{7x}{2}\cos\frac{x}{2}\,dx$
8. $\int \cos^7 2x\sin 2x\,dx$
9. $\int \sin^4 3x\cos^2 3x\,dx$
10. $\int \frac{xdx}{\sqrt{6x^2-x^4-5}}$

CHAPTER 8

INTEGRATION TECHNIQUES

8–1. Introduction

The previous chapter has provided us with the rules to integrate a limited number of expressions. We call these expressions the standard forms.

This chapter is intended to enable us to transform and reduce many nonstandard forms to the standard and thus integrate a greater variety of forms. Although other techniques are available, we will limit our development to four basic techniques; integration by parts, algebraic substitution, trigonometric substitution, and partial fractions.

8–2. Integration by Parts

The rule for finding the differential of the product of two functions, u and v, is

$$d(uv) = udv + vdu.$$

Rearranging we obtain

$$udv = d(uv) - vdu.$$

We integrate this last expression;

$$\int udv = \int d(uv) - \int vdu$$

and since

$$\int df = f, \int d(uv) = uv. \qquad \text{(f denotes a function of any variables)}$$

$$\therefore \int udv = uv - \int vdu.$$

This is the fundamental formula for integration by parts. (We have omitted the constant of integration arising from $\int d(uv)$ for the moment. We will include it in the constant resulting from evaluating $\int vdu$.)

The great value of the integration by parts formula arises from the fact that while we wish to evaluate the integral, $\int udv$, it may be accomplished by evaluating a different integral, $\int vdu$. If $\int udv$ is difficult to evaluate, proper selection of u and dv may result in the integral, $\int vdu$, being less difficult.

We now solve several examples to illustrate the general procedure, and then list guidelines for selecting u and dv.

Example 8–1.

Using the technique of integration by parts, evaluate the following integrals.

1. $\int x \sin x \, dx$

Let $u = x$ and $dv = \sin x \, dx$. Then $du = dx$ and $v = -\cos x$. (We omit the constant of integration until the final integration is completed.)

The formula is

$$\int udv = uv - \int vdu.$$

Substituting

$$\int x \sin x \, dx = -x \cos x - \int -\cos x \, dx$$
$$= -x \cos x + \sin x + C.$$

2. $\int ln\, x \, dx$

Let $u = ln\, x$ and $dv = dx$. Then $du = \dfrac{dx}{x}$ and $v = x$. Substituting in the formula immediately gives

$$\int ln\, x \, dx = x \, ln\, x - \int x \frac{dx}{x}.$$
$$= x \, ln\, x - x + C$$
$$= x\,(ln\, x - 1) + C$$

3. $\int x \, ln\, x \, dx$

Let $u = ln\, x$ and $dv = x\, dx$. Then $du = \dfrac{dx}{x}$ and $v = \dfrac{x^2}{2}$.

$$\int x \ln x\, dx = \frac{x^2 \ln x}{2} - \int \frac{x^2}{2}\frac{dx}{x}$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$$

$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + C$$

4. $\int \sec^3 t\, dt$

Let $u = \sec t$ and $dv = \sec^2 t\, dt$. Then $du = \sec t \tan t\, dt$ and $v = \tan t$.

$$\int \sec^3 t\, dt = \sec t \tan t - \int \sec t \tan^2 t\, dt$$

But $\tan^2 t = \sec^2 t - 1$ so that

$$\int \sec^3 t\, dt = \sec t \tan t - \int \sec t(\sec^2 t - 1)dt$$

$$= \sec t \tan t - \int \sec^3 t\, dt + \int \sec t\, dt$$

Solve the equation for $\int \sec^3 t\, at$;

$$2\int \sec^3 t\, dt = \sec t \tan t + \ln(\sec t + \tan t) + C_1$$

$$\int \sec^3 t\, dt = \frac{1}{2}[\sec t \tan t + \ln(\sec t + \tan t)] + C$$

Where $\frac{C_1}{2} = C$.

5. $\int x e^{bx} dx$

Let $u = x$ and $dv = e^{bx}\, dx$. Then $du = dx$ and $v = \frac{e^{bx}}{b}$.

$$\int x e^{bx} dx = \frac{x e^{bx}}{b} - \int \frac{e^{bx}}{b} dx$$

$$= \frac{x e^{bx}}{b} - \frac{e^{bx}}{b^2} + C$$

$$= \frac{e^{bx}}{b}\left(x - \frac{1}{b}\right) + C$$

6. $\int e^{bx} \cos nx\, dx$

Let $u = e^{bx}$ and $dv = \cos nx\, dx$. Then $du = be^{bx}dx$ and $v = \frac{1}{n}\sin nx$.

A. $\int e^{bx} \cos nx dx = \frac{1}{n} e^{bx} \sin nx - \int \frac{b}{n} e^{bx} \sin nx\, dx$

Factor out $\frac{b}{n}$ from the last integral and evaluate this integral in turn by parts.
Let $u = e^{bx}$ and $dv = \sin nx\, dx$. Then $du = be^{bx}dx$ and $v = -\frac{1}{n}\cos nx$.

B. $\int e^{bx} \sin nx\, dx = -\frac{1}{n} e^{bx} \cos nx - \int -\frac{b}{n} e^{bx} \cos nx\, dx$

Substitute this result into our first integration, equation A;

$$\int e^{bx} \cos nx\, dx = \frac{e^{bx}}{n} \sin nx - \frac{b}{n}\left[e^{bx}\left(-\frac{1}{n}\cos nx\right) - \left(-\frac{b}{n} e^{bx} \cos nx\, dx\right)\right]$$

$$= \frac{e^{bx}}{n} \sin nx + \frac{b}{n^2} e^{bx} \cos nx + \frac{b}{n}\int\left(-\frac{b}{n} e^{bx} \cos nx\, dx\right)$$

$$= \frac{e^{bx}}{n} \sin nx + \frac{b}{n^2} e^{bx} \cos nx - \frac{b^2}{n^2}\int e^{bx} \cos nx\, dx$$

Transpose $-\frac{b^2}{n^2}\int e^{bx} \cos nx\, dx$ and factor $\int e^{bx} \cos nx\, dx$:

$$\int e^{bx} \cos nx\, dx + \frac{b^2}{n^2}\int e^{bx} \cos nx\, dx = \frac{e^{bx}}{n} \sin nx + \frac{be^{bx}}{n^2} \cos nx$$

$$\left(1 + \frac{b^2}{n^2}\right) e^{bx} \cos nx\, dx = \frac{e^{bx}}{n} \sin nx + \frac{be^{bx}}{n^2} \cos nx + C$$

$$\int e^{bx} \cos nx\, dx = \frac{\frac{e^{bx}}{n}\left(\sin nx + \frac{b}{n}\cos nx\right)}{1 + \frac{b^2}{n^2}} + C$$

$$=\frac{\frac{e^{bx}}{n}\left(\sin nx+\frac{b}{n}\cos nx\right)}{\frac{n^2+b^2}{n^2}}+C$$

$$=\frac{ne^{bx}\left(\sin nx+\frac{b}{n}\cos nx\right)}{n^2+b^2}+C$$

$$=\frac{e^{bx}(n\sin nx+b\cos nx)}{n^2+b^2}+C$$

In each of the preceding problems, the integration depended upon the proper selection of u and dv. The following suggestions should be of assistance in selecting u and dv, although the particular problem may cause you to vary your choice.

a. Since we must integrate dv to obtain v, the selection for dv must be more easily integrable.

b. The differential dx, must be contained in the expression for dv.

c. Select u so that du is a simpler expression.

8-3. Integration by Algebraic Substitution

In many problems integrals containing the combination $(a+bx^m)^{\frac{p}{q}}$ will occur. Further difficulties occur if this combination is multiplied by a power of x. The general form is $\int x^n(a+bx^m)^{\frac{p}{q}}\,dx$ with p and q restricted to integers.

The substitution $z=(a+bx^m)^{\frac{1}{q}}$ will generally reduce the problem to a more readily integrable form. The following examples will clarify the discussion.

Example 8-2.

Evaluate the following integrals using algebraic substitution.

1. $\int x(3+x)^{1/2}\,dx$

Comparing this problem with the general form we see that $n=1$, $m=1$, $p=1$, and $q=2$.

Let $z=(3+x)^{1/2}$ from which $x=z^2-3$ and $dx=2zdz$.

Substituting in the original expression,

$$\therefore\int x(3+x)^{1/2}\,dx=\int(z^2-3)(z)\,2zdz$$

$$=\int(2z^4-6z^2)dz$$

$$=\frac{2z^5}{5}-2z^3+C$$

Replace z by its equivalent $(3+x)^{1/2}$.

$$\therefore\int x(3+x)^{1/2}\,dx=\frac{2}{5}(3+x)^{5/2}-2(3+x)^{3/2}+C$$

2. $\int x^3(4+x^2)^{2/3}\,dx$

Let $z=(4+x^2)^{1/3}$ so that $(4+x^2)^{2/3}=z^2$.

Solve for x;

$$x^2=z^3-4 \text{ and}$$
$$x=(z^3-4)^{1/2}$$

From this last equation we obtain

$$x^3=(z^3-4)^{3/2}$$

Differentiating the expression $x^2=z^3-4$,

$$2xdx=3z^2dz$$

$$dx=\frac{3z^2dz}{2x}$$

Substituting $x=(z^3-4)^{1/2}$ in the above equation,

$$dx=\frac{3z^2\,dz}{2(z^3-4)^{1/2}}$$

Substituting $x^3=(z^3-4)^{3/2}$, $(4+x^2)^{2/3}=z^2$ and $dx=\frac{3z^2\,dz}{2(z^3-4)^{1/2}}$ in the original integral,

$$\int x^3(4+x^2)^{2/3}\,dx=\int(z^3-4)^{3/2}(z^2)\frac{3z^2\,dz}{2(z^3-4)^{1/2}}$$

$$=\frac{1}{2}\int(z^3-4)(z^2)\,3z^2\,dz$$

$$=\frac{1}{2}\int(3z^7-12z^4)dz$$

$$=\frac{1}{2}\left(\frac{3z^8}{8}-\frac{12z^5}{5}\right)+C.$$

But $z=(4+x^2)^{1/3}$ so that

$$\int x^3(4+x^2)^{2/3}\,dx=\frac{3(4+x^2)^{8/3}}{16}-\frac{6(4+x^2)^{5/3}}{5}+C$$

3. $\int\frac{x^2dx}{(3+x)^{3/4}}$

Let $z=(3+x)^{1/4}$ and $z^3=(3+x)^{3/4}$.

Solve for x;

$$x = z^4 - 3 \text{ and}$$
$$x^2 = (z^4-3)^2$$
$$dx = 4z^3 dz.$$

$$\therefore \int \frac{x^2 dz}{(3+x)^{3/4}} = \int \frac{(z^4-3)^2\, 4z^3 dz}{z^3}$$
$$= \int 4(z^4-3)^2\, dz$$
$$= \int 4\,(z^8 - 6z^4 + 9)\, dz$$
$$= 4\left(\frac{z^9}{9} - \frac{6z^5}{5} + 9z\right) + C$$

and since $z = (3+x)^{1/4}$, we have

$$\int \frac{x^2 dx}{(3+x)^{3/4}} = 4\left[\frac{(3+x)^{9/4}}{9} - \frac{6(3+x)^{5/4}}{5} + 9(3+x)^{1/4}\right] + C$$

8–4. Integration by Trigonometric Substitution

When radicals of the form $\sqrt{a^2-u^2}$ or $\sqrt{u^2 \pm a^2}$ appear, we may usually simplify the integral by trigonometric substitution. In most cases the proper substitution suggested below will remove the radical and thus enable us to integrate.

1. If $\sqrt{a^2-u^2}$ appears, substitute $u = a \sin \theta$. $\sqrt{a^2-u^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = a\sqrt{1-\sin^2\theta} = a \cos \theta$
2. If $\sqrt{u^2+a^2}$ appears, substitute $u = a \tan \theta$. $\sqrt{u^2+a^2} = \sqrt{a^2 \tan^2\theta + a^2} = a\sqrt{\tan^2\theta + 1} = a \sec \theta$
3. If $\sqrt{u^2-a^2}$ appears, substitute $u = a \sec \theta$. $\sqrt{u^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\sqrt{\sec^2\theta - 1} = a \tan \theta$

The trigonometric substitution method of solving integrals is simplified if we accompany our substitution with a right triangle. For instance, suppose we let $u = a \sin \theta$. We may rearrange this equation to yield

$$\sin \theta = \frac{u}{a}.$$

Figure 8–1 is the triangle with $\sin \theta = \frac{u}{a}$. The remaining side is labeled $\sqrt{a^2-u^2}$. The following example will indicate the utilization of the triangle.

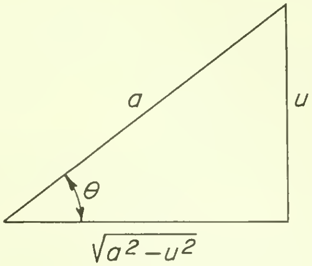

Figure 8–1.—$\sin \theta = \frac{u}{a}$.

Example 8–3.

Integrate the given problems using trigonometric substitution.

1. $\displaystyle\int \frac{x^2\, dx}{\sqrt{4-9x^2}}$

We have $a=2$ and $u=3x$ so that the substitution $3x = 2 \sin \theta$ is appropriate. Hence $x = \frac{2}{3} \sin \theta$, $x^2 = \frac{4}{9}\sin^2\theta$, and $dx = \frac{2}{3}\cos\theta\, d\theta$.

The accompanying triangle with all sides properly labeled is shown in figure 8–2.

$$\therefore \int \frac{x^2 dx}{\sqrt{4-9x^2}} = \int \frac{\frac{4}{9}\sin^2\theta \frac{2}{3}\cos\theta d\theta}{\sqrt{4-4\sin^2\theta}}$$
$$= \frac{4}{27}\int \frac{\sin^2\theta \cos\theta d\theta}{\sqrt{\cos^2\theta}}$$
$$= \frac{4}{27}\int \sin^2\theta d\theta \text{ and since}$$

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta, \text{ we have}$$

$$\int \frac{x^2 dx}{\sqrt{4-9x^2}} = \frac{4}{27}\int \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$
$$= \frac{4}{27}\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right) + C$$

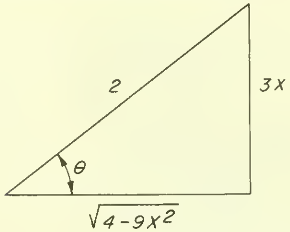

Figure 8–2.—$\theta = \sin^{-1} \frac{3x}{2}$.

We see from figure 8–2 that θ is the angle whose sine is $\frac{3x}{2}$ or $\theta = \sin^{-1} \frac{3x}{2}$. Identities give us $\sin 2\theta = 2 \sin \theta \cos \theta$ and again from the figure we see $\sin \theta = \frac{3x}{2}$ and $\cos \theta = \frac{\sqrt{4-9x^2}}{2}$.

Finally,

$$\int \frac{x^2 dx}{\sqrt{4-9x^2}} = \frac{4}{27}\left[\frac{1}{2} \sin^{-1} \frac{3x}{2} - \left(\frac{1}{4}\right)\left(\frac{6x}{2}\right)\left(\frac{\sqrt{4-9x^2}}{2}\right)\right]$$

$$= \frac{2}{27}\left(\sin^{-1} \frac{3x}{2} - \frac{3x}{4}\sqrt{4-9^2}\right)$$

The use of the accompanying figure (triangle) is well illustrated by this problem.

2. $\int \sqrt{3+4x^2}\ dx$

Examination shows that $a = \sqrt{3}$ and $u = 2x$. The proper substitution is $2x = \sqrt{3} \tan \theta$. Figure 8–3 is the accompanying triangle. For substitution into the original integral we have

$$x = \frac{\sqrt{3}}{2} \tan \theta \text{ and}$$

$$dx = \frac{\sqrt{3}}{2} \sec^2 \theta\ d\theta.$$

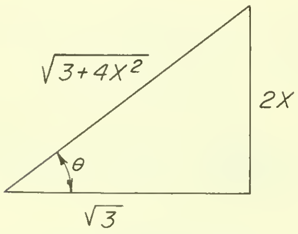

Figure 8–3.—Sec $\theta = \frac{\sqrt{3+4x^2}}{\sqrt{3}}$.

$$\therefore \int \frac{1}{\sqrt{3+4x^2}} dx = \int \frac{\frac{\sqrt{3}}{2} \sec^2 \theta\ d\theta}{\sqrt{3+3\tan^2 \theta}}$$

$$= \int \frac{\frac{\sqrt{3}}{2} \sec^2 \theta\ d\theta}{\sqrt{3} \sec \theta}$$

$$= \frac{1}{2} \int \sec \theta\ d\theta$$

This is of the form of formula (16) chapter 7:

$$\int \sec u = \ln(\sec u + \tan u) + C$$

$$\therefore \frac{1}{2} \int \sec \theta\ d\theta = \frac{1}{2} \ln(\sec \theta + \tan \theta) + C$$

From figure 8–3 we find that

$$\sec \theta = \frac{1}{\cos \theta} = \frac{\sqrt{3+4x^2}}{\sqrt{3}}$$

$$\tan \theta = \frac{2x}{\sqrt{3}}$$

Finally

$$\int \frac{1}{\sqrt{3+4x^2}} dx = \frac{1}{2} \ln\left(\frac{\sqrt{3+4x^2}}{\sqrt{3}} + \frac{2x}{\sqrt{3}}\right) + C$$

3. $\int \frac{dx}{x\sqrt{16x-9}}$

Inspection reveals $a = 3$ and $u = 4\sqrt{x}$. We let $4\sqrt{x} = 3 \sec \theta$ and the triangle illustrating this is given in figure 8–4.

$$\sqrt{x} = \frac{3}{4} \sec \theta$$

$$x = \frac{9}{16} \sec^2 \theta \text{ so that}$$

$$dx = \frac{9}{8} \sec \theta \sec \theta \tan \theta d\theta \text{ or}$$

$$dx = \frac{9}{8} \sec^2 \theta \tan \theta d\theta.$$

$$\therefore \int \frac{dx}{x\sqrt{16x-9}} = \int \frac{\frac{9}{8} \sec^2 \theta \tan \theta d\theta}{\frac{9}{16} \sec^2 \theta \sqrt{9 \sec^2 \theta - 9}}$$

$$= \int \frac{2 \tan \theta d\theta}{3\sqrt{\sec^2 \theta - 1}} = \frac{2}{3} \int d\theta$$

$$= \frac{2}{3} \theta + C$$

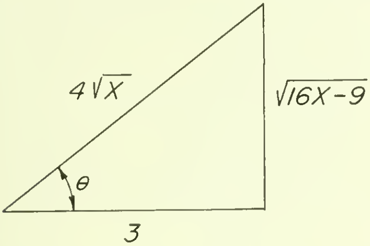

Figure 8–4.—$\theta = \sec^{-1} \frac{4\sqrt{x}}{3}$.

Figure 8–4 tells us that $\theta = \sec^{-1} \frac{4\sqrt{x}}{3}$.

$$\therefore \int \frac{dx}{x\sqrt{16x-9}} = \frac{2}{3}\sec^{-1}\frac{4\sqrt{x}}{3} + C$$

8–5. Integration by Use of Partial Fractions

The techniques for forming partial fractions from proper or improper fractions of the form $\frac{f(x)}{g(x)}$ is thoroughly dealt with in NavPers 10071. In this particular case, $f(x)$ and $g(x)$ are polynominals in the independent variable, x. Until the student has mastered this material, it would be of little value for him to proceed with the work in this section.

Example 8–4.

Carry out the following integrations using the partial fraction resolution techniques.

1. $\int \frac{(3x-20)dx}{x^2+3x-10}$

We first factor $x^2+3x-10$ into $(x+5)(x-2)$ thus obtaining the denominator as the product of two real linear factors, both different. We now proceed to resolve the integrand into partial fractions.

$$\frac{3x-20}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2} \text{ so that}$$

$$3x-20 = A(x-2) + B(x+5)$$

Let $x=2$, then, $-14 = 7B$ and $B = -2$.

Let $x=-5$, then, $-35 = -7A$ and $A = 5$.

$$\therefore \frac{3x-20}{(x+5)(x-2)} = \frac{5}{x+5} - \frac{2}{x-2}$$

Thus we integrate,

$$\int \frac{(3x-20)dx}{x^2+3x-10} = \int \frac{5dx}{x+5} - \int \frac{2dx}{x-2}$$
$$= 5ln(x+5) - 2ln(x-2) + C$$
$$= ln\left[\frac{(x+5)^5}{(x-2)^2}\right] + C$$

2. $\int \frac{(x^3+5x^2+3x-5)dx}{(x+1)^2(x+3)^2}$

We note that the factors in the denominator of this integrand are linear, but repeated. Resolving the integrand into partial fractions we obtain

$$\frac{x^3+5x^2+3x-5}{(x+1)^2(x+3)^2} = \frac{A}{(x+1)^2} + \frac{B}{(x+1)} + \frac{C}{(x+3)^2} + \frac{D}{(x+3)}$$

and equating numerators once the common denominator is formed on the right

$$x^3+5x^2+3x-5 = A(x+3)^2 + B(x+1)(x+3)^2 + C(x+1)^2 + D(x+3)(x+1)^2. \quad (1)$$

Multiply through and collect coefficients with like powers of x.

$$x^3+5x^2+3x-5 = (B+D)x^3 + (A+7B+C+5D)x^2 + (6A+15B+2C+7D)x + (9A+9B+C+3D) \quad (2)$$

Let $x = -1$ in equation (1) from which we get $A = -1$. Then, in the same equation we let $x = -3$ and find that $C = 1$.

On either side of equation (2) we equate coefficients of like powers of x to obtain

$$B+D = 1 \text{ and} \quad (3)$$
$$A+C+7B+5D = 5 \quad (4)$$

After substituting $A = -1$ and $C = 1$ into equation (4) we solve (3) and (4) simultaneously to obtain $B = 0$ and $D = 1$.

$$\therefore \int \frac{(x^3+5x^2+3x-5)dx}{(x+1)^2(x+3)^2} = -\int \frac{dx}{(x+1)^2} + \int \frac{dx}{(x+3)^2} + \int \frac{dx}{(x+3)}$$
$$= -\int (x+1)^{-2}dx + \int (x+3)^{-2}dx + \int \frac{dx}{(x+3)}$$
$$= -\frac{(x+1)^{-1}}{-1} + \frac{(x+3)^{-1}}{-1} + ln(x+3) + C$$
$$= \frac{1}{x+1} - \frac{1}{x+3} + ln(x+3) + C$$
$$= \frac{(x+3)-(x+1)}{(x+1)(x+3)} + ln(x+3) + C$$
$$= \frac{2}{(x+1)(x+3)} + ln(x+3) + C$$

3. $\int \frac{(9x^2+10x+26)dx}{(2x+3)(x^2+4)}$

This integrand contains a nonrepeated quadratic factor in its denominator.

$$\frac{9x^2+10x+26}{(2x+3)(x^2+4)}=\frac{A}{2x+3}+\frac{Bx+C}{x^2+4}$$

Form a common denominator on the right hand side and equate numerators.

$$9x^2+10x+26=A(x^2+4)+(Bx+C)(2x+3)$$

Multiply through and collect coefficients of like powers of x.

$$9x^2+10x+26=(A+2B)x^2+(3B+2C)x+(4A+3C)$$

Equate coefficients of like powers of x obtaining

$$A+2B=9$$

$$3B+2C=10$$

$$4A+3C=26$$

The simultaneous solution of these equations gives us $A=5$, $B=2$, and $C=2$.

Substituting these values in the original expression,

$$\frac{9x^2+10x+26}{(2x+3)(x^2+4)}=\frac{5}{2x+3}+\frac{2x+2}{x^2+4}$$

$$\therefore \int \frac{(9x^2+10x+26)dx}{(2x+3)(x^2+4)}=\int \frac{5dx}{2x+3}+\int \frac{(2x+2)dx}{x^2+4}$$

$$=5\int \frac{dx}{2x+3}+\int \frac{2xdx}{x^2+4}+\int \frac{2dx}{x^2+4}$$

The first term on the right of the equality is like formula (20) chapter 7:

$$\int \frac{du}{au+b}=\frac{1}{a}\,ln(au+b)$$

$$5\int \frac{dx}{2x+3}=(5)\frac{1}{2}\,ln(2x+3)$$

$$=\frac{5}{2}\,ln(2x+3)$$

The second term on the right of the equality is like formula (21) chapter 7:

$$\int \frac{udu}{au^2+b}=\frac{1}{2a}\,ln(au^2+b)$$

$$\int \frac{2xdx}{x^2+4}=2\int \frac{xdx}{x^2+4}=(2)\frac{1}{2}\,ln(x^2+4)$$

$$=ln(3x^2+4)$$

The third term on the right of the equality is like formula (22) chapter 7:

$$\int \frac{du}{au^2+b}=\frac{1}{\sqrt{ab}}\tan^{-1}\left(u\sqrt{\frac{a}{b}}\right)$$

$$2\int \frac{dx}{x^2+4}=\frac{(2)1}{\sqrt{(1)(4)}}\tan^{-1}\left(x\sqrt{\frac{1}{4}}\right)$$

$$=\tan^{-1}\frac{x}{2}$$

$$\therefore \int \frac{(9x^2+10x+26)dx}{(2x+3)(x^2+4)}=\frac{5}{2}\,ln(2x+3)+ln(x^2+4)+\tan^{-1}\frac{x}{2}+C$$

$$=ln\left[(2x+3)^{\frac{5}{2}}(x^2+4)\right]+\tan^{-1}\frac{x}{2}+C$$

4. $\int \frac{(t^5+4t^3)dt}{(t^2+2)^3}$

The denominator of the integrand contains a repeated quadratic factor.

$$\frac{t^5+4t^3}{(t^2+2)^3}=\frac{At+B}{(t^2+2)^3}+\frac{Ct+D}{(t^2+2)^2}+\frac{Et+F}{(t^2+2)}$$

Equate numerators once the common denominator is formed on the right.

$$t^5+4t^3=(At+B)+(Ct+D)(t^2+2)+(Et+F)(t^2+2)^2$$

Multiply out and collect coefficients of like powers of t.

$$t^5+4t^3=Et^5+Ft^4+(C+4E)t^3+(D+4F)t^2$$

$$+(A+2C+4E)t+(B+2D+4F)$$

Equate coefficients of like powers of t.

$$E=1$$

$$F=0$$

$$C+4E=4$$

$$D+4F=0$$

$$A+2C+4E=0$$

$$B+2D+4F=0$$

Thus, $E=1$, $A=-4$, and $B=C=D=F=0$. Substituting these values in the original expression,

$$\frac{t^5+4t^3}{(t^2+2)^3}=\frac{-4t+0}{(t^2+2)^3}+\frac{0t+0}{(t^2+2)^2}+\frac{t+0}{t^2+2}$$

$$=\frac{-4t}{(t^2+2)^3}+\frac{t}{t^2+2}$$

$$\therefore \int\frac{(t^5+4t^3)dt}{(t^2+2)^3}=\int\frac{tdt}{t^2+2}-4\int\frac{tdt}{(t^2+2)^3}\cdot$$

The first term on the right of the equality is similar to formula (21) chapter 7:

$$\int\frac{udu}{au^2+b}=\frac{1}{2a}\,ln(au^2+b)$$

$$\int\frac{tdt}{t^2+2}=\frac{1}{2}\,ln(t_1^2+2)$$

The last term on the right is like formula (23) chapter 7:

$$\int(au^2+b)^n udu=\frac{1}{2a}\,\frac{(au^2+b)^{n+1}}{n+1}\,;\,n\neq-1$$

$$-4\int\frac{tdt}{(t^2+2)^3}=-4\int(t^2+2)^{-3}tdt$$

$$=-4\,\frac{1}{2}\cdot\frac{(t+2)}{-3+1}$$

$$=\frac{1}{(t^2+2)^2}$$

$$\therefore \int\frac{t^5+4t^3}{(t^2+2)^3}=\frac{1}{2}\,ln(t^2+2)+\frac{1}{(t^2+2)^2}+C$$

$$=ln(t^2+2)^{1/2}+\frac{1}{(t^2+2)^2}+C$$

Exercise 8–1.

1. $\int x\cos x dx$
2. $\int\frac{dx}{x^2\sqrt{4-x^2}}$
3. $\int\frac{(4t-2)dt}{t^3-t^2-2t}$
4. $\int e^{bx}\sin nx dx$
5. $\int x\sqrt{x-4}dx$
6. $\int\frac{dv}{(v^2-9)^{3/2}}$

8–6 Improper Integrals

The integrals with which we have been concerned have been proper integrals. We will now investigate the properties of the IMPROPER INTEGRAL. The definite integral, $\int_a^b f(x)dx$, may become improper in two ways;

(1) EITHER or BOTH of the limits of integration become infinite or

(2) the integrand, $f(x)$, becomes infinite at point a, or at point b, or any point in between these end points of the interval from a to b.

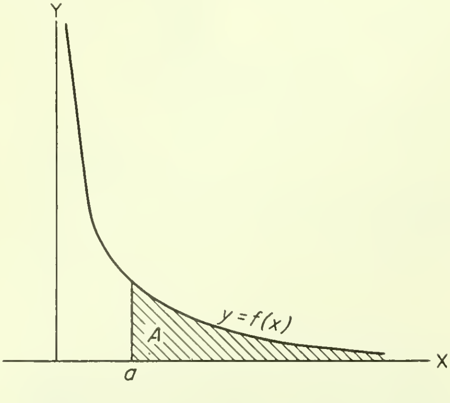

Figure 8-5.—$A=\int_a^\infty f(x)dx=\lim_{b\to\infty}\int_a^b f(x)dx$ provided the limit exists.

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We will handle each of the above cases separately.

Case (1): The limits of integration become infinite.

A. The upper limit is infinite. We then use the following definition.

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

The definition is valid provided the limit exists. If the limit on the right does not exist, the integral does not exist. Figure 8–5 illustrates how this improper integral may be interpreted as an area.

B. The lower limit is infinite. By definition

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$$

provided the limit exists. Figure 8–6 provides an area interpretation for this integral.

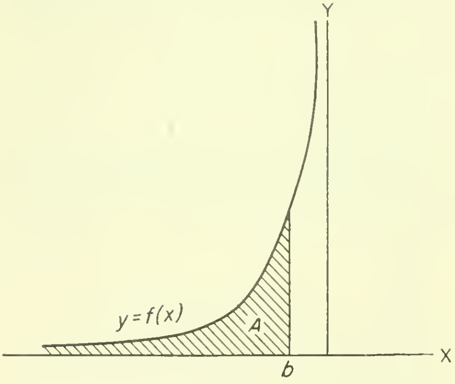

Figure 8–6. —$A=\int_{-\infty}^{b} f(x)dx = \lim_{a\to\infty}\int_a^b f(x)dx.$

C. Both the upper and lower limits are infinite.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx$$

Note here that we have split the original integral into two integrals at the point c (fig. 8–7). Point c may be any convenient finite point. Each of the integrals may now be handled as in part A or part

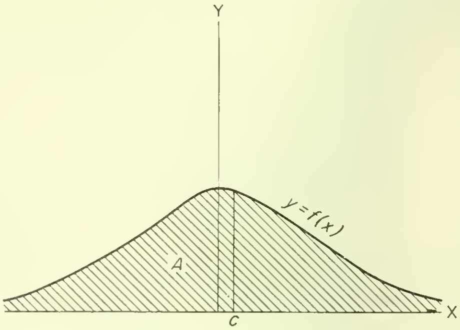

Figure 8–7.—$A=\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_a^c f(x)dx + \lim_{b\to\infty}\int_c^b f(x)dx$ provided both limits exist.

B. That is,

$$\int_{-\infty}^{c} f(x)dx = \lim_{a\to-\infty}\int_a^c f(x)dx \text{ and}$$

$$\int_c^{\infty} f(x)dx = \lim_{b\to\infty}\int_c^b f(x)dx$$

BOTH of these limits must exist in order for the integral to exist. If either limit does not exist, the integral does not exist.

Case (2): The integrand becomes infinite or discontinuous either at the limits of integration themselves, or some point in the interval between them.

Figure 8–8 shows $f(x)$ with a discontinuity at point p. We wish to calculate the area indicated on the figure, but we must learn how to deal with function at point p.

The area WITHOUT the discontinuity would be given by

$$\int_a^b f(x)dx.$$

The area from a to p is by definition

$$\int_a^p f(x)dx = \lim_{t\to p-}\int_a^t f(x)dx.$$

We recall from section 5–4 that we called this the left-hand limit of the function at point p. We take the limit of the function as t approaches p from the left.

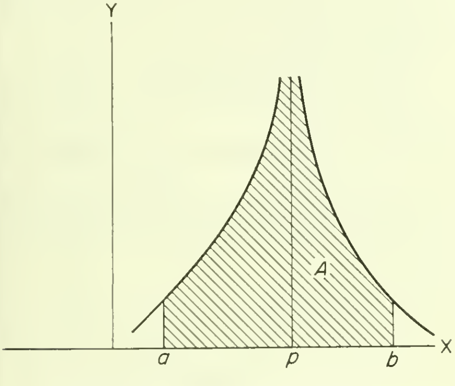

Figure 8–8.—Plot of $f(x)$ with discontinuity at P.

Similarly, the area from p to b is given by

$$\int_p^b f(x)dx = \lim_{t\to p+}\int_t^b f(x)dx.$$

This notation indicates the right-hand limit of the function as t approaches p from the right. Again, these limits must exist in order for the integral to exist.

The total area is then

$$A=\int_a^b f(x)dx=\int_a^p f(x)dx+\int_P^b f(x)dx$$

$$=\lim_{t\to p-}\int_a^t f(x)dx+\lim_{t\to p+}\int_t^b f(x)dx$$

When working problems, care must be taken to prevent integrating over an interval that contains a section where the integrand is not continuous.

Example 8–5.

Evaluate the following integrals.

1. $\displaystyle\int_3^\infty \frac{dx}{x^4}=\lim_{b\to\infty}\int_3^b\frac{dx}{x^4}=\lim_{b\to\infty}\left.\frac{1}{-3x^3}\right|_3^b$

from formula (4) chapter 7.

$$=\lim_{b\to\infty}\left[\frac{1}{-3b^3}+\frac{1}{-3(3)^4}\right]$$

$$=\lim_{b\to\infty}\left[\frac{1}{-3b^3}+\frac{1}{(3)^4}\right]$$

$$=\frac{1}{81}$$

Since the $\displaystyle\lim_{b\to\infty}\frac{1}{-3b^3}=0.$

2. $\displaystyle\int_0^\infty\frac{x\,dx}{x^2+1}=\lim_{b\to\infty}\int_0^b\frac{x\,dx}{x^2+1}=\lim_{b\to\infty}\left.\left(\frac{1}{2}ln\,(x^2+1)\right)\right|_0^b$

from formula (21) chapter 7.

$$=\lim_{b\to\infty}\frac{1}{2}ln(b^2+1)-\frac{1}{2}ln\ 1$$

But $ln\infty$ is unbounded so that the limit does not exist. Therefore, the integral does not exist.

3. $\displaystyle\int_0^4\frac{dx}{(x-3)^2}$

When $x=3$, the integrand, $\dfrac{1}{(x-3)^2}$, has an infinite discontinuity. We must split the original integral into two integrals at $x=3$.

$$\therefore\int_0^4\frac{dx}{(x-3)^2}=\int_0^3\frac{dx}{(x-3)^2}+\int_3^4\frac{dx}{(x-3)^2}$$

$$=\lim_{t\to3-}\int_0^t\frac{dx}{(x-3)^2}+\lim_{t\to3+}\int_t^4\frac{dx}{(x-3)^2}$$

$$=\lim_{t\to3-}\left.\left(-\frac{1}{x-3}\right)\right|_0^t+\lim_{t\to3+}\left.\left(-\frac{1}{x-3}\right)\right|_t^4$$

(from formula (25) chapter 7.)

$$=\lim_{t\to3-}\left(\frac{-1}{t-3}-\frac{-1}{0-3}\right)$$

$$+\lim_{t\to3+}\left(-\frac{1}{4-3}-\frac{-1}{t-3}\right)$$

$$=\lim_{t\to3-}\left(-\frac{1}{t-3}-\frac{1}{3}\right)+\lim_{t\to3+}\left(-1+\frac{1}{t-3}\right)$$

Both of these limits fail to exist and, therefore, the integral fails to exist.

Had we ignored the discontinuity and integrated over the given limits we would have obtained the incorrect result of $-1\frac{1}{3}$.

Example 8–5.

Set up the integration required to evaluate the area beneath the step function shown shaded in figure 8–9.

The discontinuity occurs at $x=c$, and in this case it is a finite discontinuity. Thus, the taking of

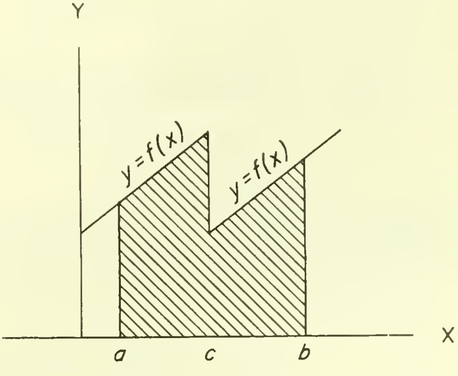

Figure 8-9.—Area under the step function $y=f(x)$.

a limit is not required. We set up the integrals as follows;

$$\text{Area}=\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx.$$

Example 8-6.

Set up the integrals necessary to evaluate the shaded area indicated in figure 8-10. The function plotted is $\frac{1}{x(x-1)}$.

First note that the limits of integration extend from $-\infty$ to $+\infty$. Also, the integral has discontinuities at $x=0$ and $x=1$. We have four reasons for calling $\int_{-\infty}^{\infty}\frac{dx}{x(x-1)}$ an improper integral. Any one of these conditions is reason enough to call the integral improper.

(1) The lower limit is infinite.

(2) The upper limit is infinite.

(3) The integrand becomes unbounded (infinite) at $x=0$.

(4) The integrand becomes unbounded at $x=1$.

It is necessary to break the original integral into six separate integrals in order to calculate the required area.

$$\int_{-\infty}^{\infty}\frac{dx}{x(x-1)}=\int_{-\infty}^{-1}\frac{dx}{x(x-1)}$$
$$+\int_{-1}^{0}\frac{dx}{x(x-1)}+\int_{0}^{1/2}\frac{dx}{x(x-1)}\int_{1/2}^{1}\frac{dx}{x(x-1)}$$
$$+\int_{1}^{2}\frac{dx}{x(x-1)}+\int_{2}^{\infty}\frac{dx}{x(x-1)}$$

Where the points $x=-1$, $\frac{1}{2}$, and 2, are convenient finite points falling in the intervals $-\infty<x<0$, $0<x<1$, and $1<x<\infty$. Each of the six integrals contains only one quality of an improper integral. The first integral is improper since its lower limit is infinite, the second integral contains an integrand that becomes unbounded at $x=0$, etc., for each integral that follows.

We employ the limit techniques just developed to each integral in turn.

$$\int_{-\infty}^{\infty}\frac{dx}{x(x-1)}=\lim_{a\to-\infty}\int_a^{-1}\frac{dx}{x(x-1)}+\lim_{t\to0-}\int_{-1}^{t}\frac{dx}{x(x-1)}$$
$$+\lim_{t\to0+}\int_t^{1/2}\frac{dx}{x(x-1)}+\lim_{t\to1-}\int_{1/2}^{t}\frac{dx}{x(x-1)}$$
$$+\lim_{t\to1+}\int_t^{2}\frac{dx}{x(x-1)}+\lim_{b\to+\infty}\int_2^{b}\frac{dx}{x(x-1)}$$

If ANY of the six limits so calculated does not exist, the original integral does not exist.

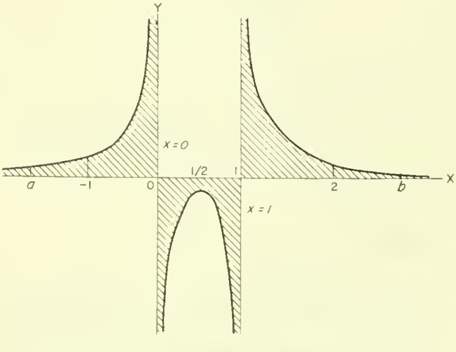

Figure 8-10.—Area under the function $y=\frac{1}{x(x-1)}$.

8-7. Determining Areas in Rectangular Coordinates

We have described the integral in previous applications as representing an area under a curve. We will develop this idea further in this section and learn how to perform actual area evaluations. The methods we intend to use are less theoretical than may be found in a more advanced calculus course. Nevertheless, they will prove adequate for our purposes.

Figure 8-11 is the plot of $y=f(x)$ which is at all points positive throughout the interval $a\leq x\leq b$.

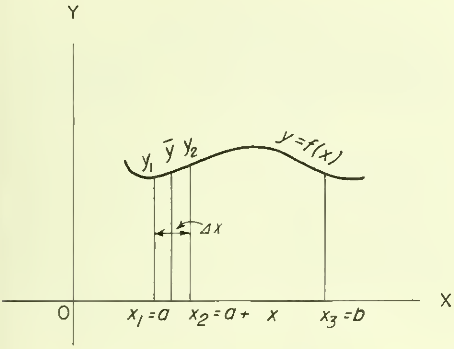

Figure 8–11.—Area under $y=f(x)$ between limits a and b.

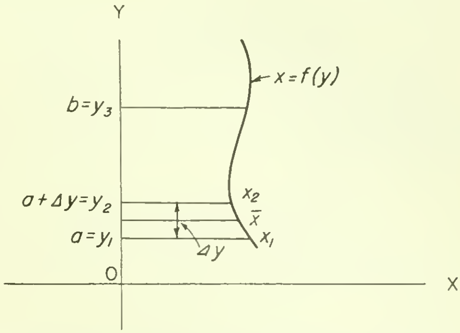

Figure 8–12.—Area bounded by $x=f(y)$ between limits a and b.

The area which we wish to determine is bounded by $y=f(x)$, $y=0$, $x_1=a$, and $x_3=b$. We have indicated on the figure a narrow rectangle of width Δx. In order to calculate the area of this small rectangle, we need its length. We choose the average length of the rectangle determined from $\frac{y_1+y_2}{2}=\bar{y}$. Therefore, the area ($\Delta A$) of this small rectangle is $\Delta A=\bar{y}\Delta x$.

Then,

$$\frac{\Delta A}{\Delta x}=\bar{y}.$$

The change of area over an interval Δx, is equal to the average ordinate (y) in that interval. Moreover, in the limit as $\Delta x \rightarrow 0$, we see from the figure that $\bar{y} \rightarrow y_1=f(x_1)=f(a)$.

$$\therefore \frac{dA}{dx}=\lim_{\Delta x\rightarrow 0}\frac{\Delta A}{\Delta x}=y \quad \text{and } dA=ydx.$$

If we sum these small areas over the entire interval from a to b, we obtain

$$A=\int_a^b dA=\int_a^b ydx=\int_a^b f(x)dx$$

You may choose to view this last integral as causing the rectangle of constant width dx, to slide along the x axis between the points a and b. As it slides, the top either stretches or contracts to fit the contour of $y=f(x)$. The area thus traced out is the area required.

Figure 8–12 is a plot of $x=f(y)$. In this case our rectangle is of average length, $\bar{x}$, in the interval Δy. The student should verify that the area in this case is

$$A=\int_a^b f(y)dy$$

by following the steps covered for the first case.

Example 8–7.

Calculate the area indicated in figure 8–13.

The function plotted is $y=\frac{x^2}{4}$ and since our area element (ΔA) is vertical, we use

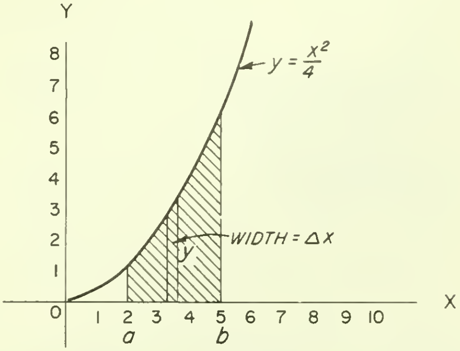

Figure 8–13.—Area under $y=\frac{x^2}{4}$ between limits $a=2$ and $b=5$.

$$A=\int_a^b ydx=\int_a^b f(x)dx.$$

Note on the figure that $y\Delta x$ is the area (ΔA) of the small rectangle shown. In this problem $f(x) = \frac{x^2}{4}$, $a=2$, and $b=5$.

$$\therefore A=\int_2^5 \frac{x^2}{4}dx=\frac{1}{4}\int_2^5 x^2dx=\frac{1}{4}\cdot\frac{x^3}{3}\Big|_2^5$$

from formula 4 chapter 7 where $n=2$, $u=x$, and $du=dx$.

$$A=\frac{5^3}{12}-\frac{2^3}{12}=\frac{125-8}{12}=\frac{117}{12}=\frac{39}{4}=9.75 \text{ square units.}$$

Example 8-8.

Solve for the inverse function of the previous problem and determine the area shaded in figure 8-14.

The direct function in the previous problem is $y = \frac{x^2}{4}$. The inverse function (see example 5-19, section 5-11, chapter 5) is $x=\pm 2y^{1/2}$.

We have shown only the plot of the positive root, $+2y^{1/2}$. The area element is horizontal and of width Δy and length x. The area is given by

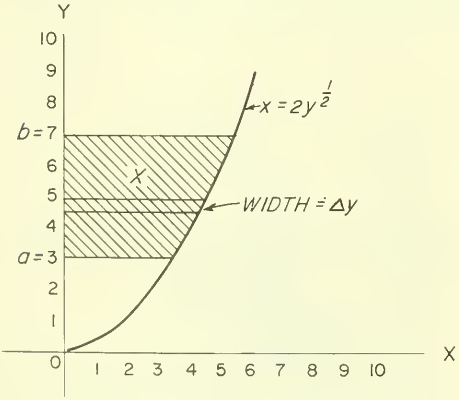

Figure 8-14.—Area bounded by inverse function of $y=\frac{x^2}{4}$ between limits $a=3$ and $b=7$.

$$A=\int_a^b xdy=\int_a^b f(y)dy.$$

The function is $f(y)=2y^{1/2}$ and the limits are $a=3$, and $b=7$.

$$\therefore A=\int_3^7 2y^{1/2}dy=2\int_3^7 y^{1/2}dy=2\frac{y^{3/2}}{\frac{3}{2}}\Bigg|_3^7=\frac{4}{3}y^{3/2}\Bigg|_3^7$$

from formula (4) chapter 7 when $n=\frac{1}{2}$, $u=y$ and $du=dy$.

$$A=\frac{4}{3}\cdot 7^{3/2}-\frac{4}{3}\cdot 3^{3/2}$$

$$=\frac{4}{3}(7^{3/2}-3^{3/2})=1.33(18.5-5.19)=(1.33)(13.3)$$

$$=17.8 \text{ square units (approx)}$$

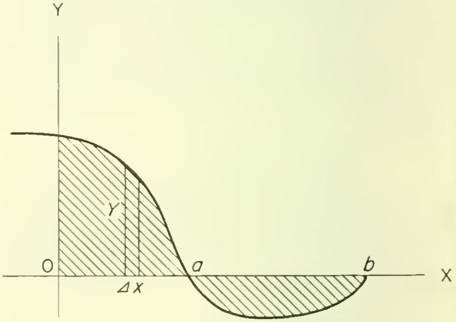

Figure 8-15.—$A=\int_0^b ydx=\int_0^b f(x)dx.$

We now look at the function plotted in figure 8-15. We use vertical area elements to evaluate the shaded area. Thus,

$$A=\int_0^b ydx=\int_0^b f(x)dx.$$

But this single integral will give us the algebraic sum of the two portions shown which comprise the total area. Note in the first portion that y is above the x axis and positive while in the second portion, y

is below the x axis and negative. Looking at the integral, we see that this will make the first portion of the area a positive quantity and the second portion a negative quantity. Thus, integrating across the interval from 0 to b with one integral will yield the algebraic sum of these two portions.

Normally we are concerned with the total AMOUNT of area, and not the algebraic sum. We therefore use the absolute quantity notation and split the original integral into two integrals. The absolute signs merely indicate that we are interested only in the MAGNITUDE of the enclosed expression and NOT THE SIGN. Thus,

$$A_T = \left| \int_0^a f(x)dx \right| + \left| \int_a^b f(x)dx \right| .$$

The result will now be the sum of the two shaded portions rather than the difference which we would have obtained from the original single integral.

Example 8–9.

Calculate (A) the net (algebraic sum) area, and (B) the total area indicated in figure 8–16.

A. The algebraic area is given by integrating directly over the interval from $x=0$ to $x=5$ with one integral.

$$A = \int_0^5 ydx = \int_0^5 (2-x)dx = 2x - \frac{x^2}{2} \Big|_0^5$$

$$A = -\frac{5}{2} \text{ square units.}$$

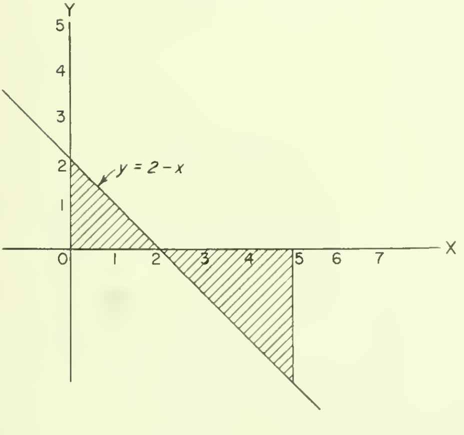

Figure 8–16.—$A = \int_0^5 ydx = \int_0^5 (2-x)dx.$

B. The total area is obtained by adding the absolute values of the two shaded portions.

$$A_T = \left| \int_0^2 ydx \right| + \left| \int_2^5 ydx \right| = \left| \int_0^2 (2-x)dx \right| + \left| \int_2^5 (2-x)dx \right|$$

$$A_T = \left| 2x - \frac{x^2}{2} \right|_0^2 + \left| 2x - \frac{x^2}{2} \right|_2^5$$

$$A_T = \left| 4-2 \right| + \left| \left(10 - \frac{25}{2}\right) - (4-2) \right|$$

$$A_T = 2 + \frac{9}{2} = 6\frac{1}{2} \text{ square units}$$

We have been concerned with areas bounded by a function and the coordinate axes. Many instances occur in solving practical problems where it becomes necessary to determine the area between two curves.

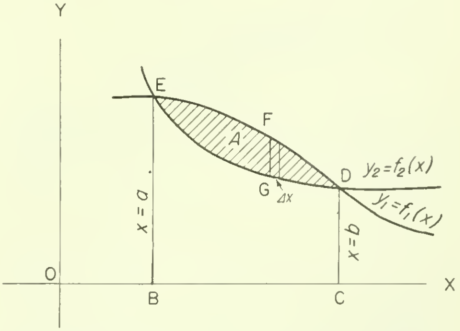

Figure 8–17.—Area bounded by $y_1 = f_1(x)$ and $y_2 = f_2(x)$.

In figure 8–17 we are interested in determining the area bounded by $y_1 = f_1(x)$ and $y_2 = f_2(x)$. We can determine this area by first calculating the area labeled BCDFE (A_1) and then subtracting from this result, the area labeled BCDGE (A_2).

First using vertical area elements,

$$A_1 = \int_a^b y_1 dx \text{ and } A_2 = \int_a^b y_2 dx.$$

$$A_1 - A_2 = \int_a^b y_1 dx - \int_a^b y_2 dx = \int_a^b (y_1 - y_2) dx$$

Therefore, the area in which we are interested is

$$A=\int_a^b (y_1-y_2)dx.$$

Note that the area element used is of height (y_1-y_2) and width Δx.

In this particular case we could have used a horizontal area element of length (x_1-x_2) and width Δy (fig. 8–18). We could accomplish this simply by solving for the inverse functions associated with y_1 and y_2 so that

$$A=\int_{y=a}^{y=b} (x_1-x_2)dy.$$

In the previous problem it did not matter whether we integrated with respect to y or x. That is, it did not matter whether we chose horizontal or vertical area elements. Such is not always the case.

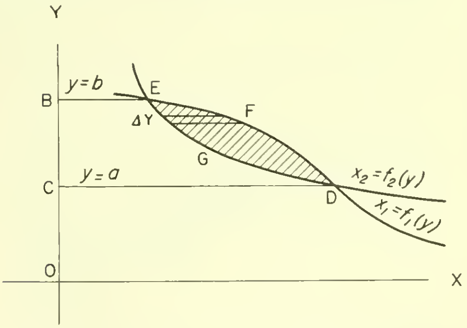

Figure 8–18.—Area bounded by $x_1=f_1(y)$ and $x_2=f_2(y)$.

The area shown in figure 8–19 requires only one integral if we choose vertical area elements (integrate with respect to x). If we choose horizontal area elements however, note that it would require two integrals. This is true, for the length of the vertical element would be expressed differently depending upon whether the element was below or above the dashed line. The length of the vertical element can be expressed as (y_1-y_2) over the entire area and thus requires only one integral.

All area calculation problems should first be sketched so as to determine two things;

(1) Is some of the area + and some − ? If so, do we wish to use absolute values?

(2) Which is the best choice of area elements?

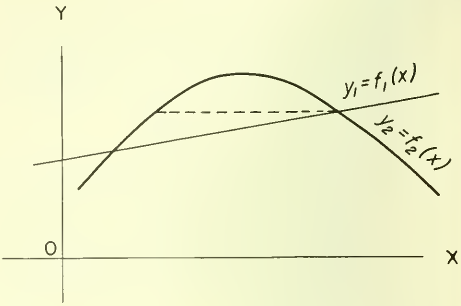

Figure 8–19.—Area bounded by $y_1=f_1(x)$ and $y_2=f_2(x)$.

Example 8–10.

Calculate the area bounded by the straight line $x=y$ and the parabola $6x=y^2$ (fig. 8–20).

An examination of the figure reveals that one integral results whether we use either vertical or horizontal area elements. We choose to use horizontal elements.

We must solve for the points of intersection in order to determine the limits of integration. Simultaneous solution of the two equations yields (0, 0) and (6, 6) as the points of intersection. The limits of y are 0 to 6.

$$A=\int_0^6 xdy=\int_0^6\left(y-\frac{y^2}{6}\right)dy=\left(\frac{y^2}{2}-\frac{y^3}{18}\right)\Bigg|_0^6$$

$$A=\frac{36}{2}-\frac{216}{18}=6 \text{ square units.}$$

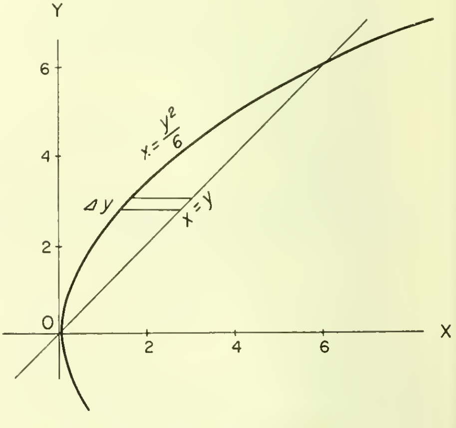

Figure 8–20.—Area bounded by $x=y$ and $6x=y^2$.

The student should verify this result by performing the integration with respect to x.

8–8. Determining Areas in Polar Coordinates

We now have a better understanding of the "area under a curve" concept. There is practically nothing new to grasp in dealing with area calculation in polar coordinates. We do need to see what the basic area element is in polar coordinates.

In rectangular coordinates we saw that the basic area element (ΔA) was either $y\Delta x$ or $x\Delta y$ depending upon whether we intended to integrate with respect to x or y. A simple ratio relationship will provide us with the basic area element in polar coordinates.

Figure 8–21 is a circle with radius ρ and containing the radian equivalent of 360° or 2π. Sketched within the circle is a sector of the circle with radius ρ and an angle of $\Delta\theta$, measured in radians. We wish to calculate the area of this sector of the circle.

Let A be the area of the circle ($\pi\rho^2$) and let ΔA be the area of the sector. Then,

$$\frac{A}{\Delta A}=\frac{2\pi}{\Delta\theta} \text{ or } \frac{\pi\rho^2}{2\pi}=\frac{\Delta A}{\Delta\theta} \text{ so that}$$

$$\frac{dA}{d\theta}=\lim_{\Delta\theta\to 0}\frac{\Delta A}{\Delta\theta}=\frac{\pi\rho^2}{2\pi}=\frac{\rho^2}{2}.$$

$\therefore$ the formula for the area element in polar coordinates is

$$dA=\frac{1}{2}\rho^2 d\theta.$$

$$\text{Hence, } A=\int_{\theta_1}^{\theta_2} dA=\frac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2 d\theta.$$

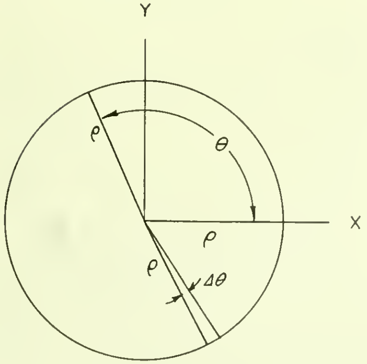

Figure 8–21.—Basic area element used with polar coordinates.

Example 8–11.

Find the area contained within $\rho=2a\cos\theta$ (fig. 8–22).

The equation when plotted is a circle with its center on the x axis. It has a radius of a so that its area is immediately πa^2. Let us check this result by integration using polar coordinates.

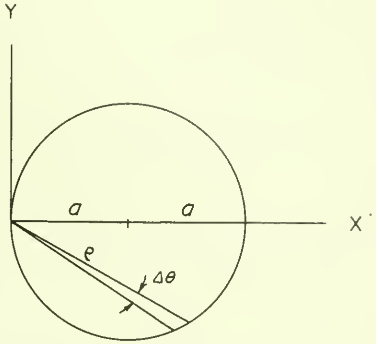

Figure 8–22.—$\rho=2a\cos\theta$.

The area element is indicated in the figure. If we measure the angle θ from the x axis, we obtain the limits of integration by setting $\rho=0$. We obtain $\theta=\frac{\pi}{2}$ and $\frac{3\pi}{2}$, but $\frac{3\pi}{2}$ is equivalent to $-\frac{\pi}{2}$ so that the limits are $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Note that these limits would permit the area element to sweep out the entire area of the circle. The same result could be achieved if we permitted the area element to sweep over the upper semicircle twice; that is, if we restrict θ to limits from 0 to $\frac{\pi}{2}$. This is true because of the symmetry involved with respect to the x axis. In equation form, then,

$$A=\frac{1}{2}\int_{-\pi/2}^{\pi/2}\rho^2 d\theta=(2)\frac{1}{2}\int_0^{\pi/2}\rho^2 d\theta$$

Now, $\rho=2a\cos\theta$ so that $A=\displaystyle\int_0^{\pi/2} 4a^2\cos^2\theta\, d\theta.$

But by an identity,

$$\cos^2 \theta = \frac{1+\cos 2\theta}{2}$$

$$A = \int_0^{\pi/2} 4a^2 \frac{(1+\cos 2\theta)}{2}\, d\theta$$

$$= \frac{4a^2}{2}\int_0^{\pi/2} d\theta + \frac{4a^2}{2}\int_0^{\pi/2} \cos 2\theta\, d\theta$$

$$= \left(2a^2\,\theta + 2a^2\,\frac{\sin 2\theta}{2}\right)\Bigg|_0^{\pi/2}$$

from formulas (3) and (26) chapter 7.

$$A = 2a^2\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_0^{\pi/2}$$

$$= 2a^2\left[\frac{\pi}{2} + \frac{\sin (2)\left(\frac{\pi}{2}\right)}{2}\right] - 2a^2\left[0 + \frac{\sin 2(0)}{2}\right]$$

$$= 2a^2\left(\frac{\pi}{2}+\frac{0}{2}\right) - 2a^2\left(0+\frac{0}{2}\right)$$

$$= \pi a^2$$

Thus, we verified our earlier result.

Example 8–12.

Determine the area enclosed within $\rho = a \sin 2\theta$ (fig. 8–23).

This figure is called a four-leaf rose. The student should take time to plot values of ρ for various values of θ.

The quadrants are labeled in the figure. From symmetry with both coordinate axes, we see that if we calculate the area of one leaf of the rose and multiply it by four, we will obtain the total area.

The limits on θ for the leaf in quadrant I are obtained by setting $\rho = 0$. When $\rho = 0$, $\theta = 0$ or $\frac{\pi}{2}$.

As mentioned previously the formula for the area element in polar coordinates is $dA = \frac{1}{2}\rho^2 d\theta$. The area, A, of the leaf in quadrant I is $A = \frac{1}{2}\int_0^{\pi/2} \rho^2 d\theta$ and the total area, A_T, for the leaves in all four quadrants is

$$A_T = (4)\frac{1}{2}\int_0^{\pi/2} \rho^2 d\theta = 2\int_0^{\pi/2} a^2 \sin^2 2\theta\, d\theta$$

Now, $\sin^2 2\theta = \frac{1}{2} - \frac{\cos 4\theta}{2}$ so that

$$A_T = 2a^2\int_0^{\pi/2}\left(\frac{1}{2} - \frac{\cos 4\theta}{2}\right) d\theta$$

$$= \frac{2a^2}{2}\int_0^{\pi/2} d\theta - \frac{2a^2}{2}\int_0^{\pi/2} \cos 4\theta\, d\theta$$

$$= (a^2\theta - a^2 \sin 4\theta)\Bigg|_0^{\pi/2}$$

from formulas (3) and (9) chapter 7.

$$A_T = a^2(\theta - \sin 4\theta)\Bigg|_0^{\pi/2}$$

$$= a^2\left(\frac{\pi}{2} - \sin\frac{4\pi}{2}\right) - a^2[0 - \sin 4(0)]$$

$$= a^2\left(\frac{\pi}{2} - 0\right) - a^2(0-0)$$

$$= \frac{\pi}{2}a^2$$

Exercise 8–2.

Find the areas bounded within the following functions. Sketch each problem before attempting a solution.

1. The lines $x+y=4$, $x=y$, and $y=0$ (the x axis).
2. The parabola $y=3x^2$ and the line $y=4$.
3. The circles $\rho = 3 \sin \theta$ and $\rho = 3 \cos \theta$.

8–9. Calculation of Work by Integration

Let us assume a situation in which a force is acting on some particle. The work that the force does on the particle is the product of the force and the distance the particle moves.

$$\text{Work} = \text{force} \times \text{distance}$$

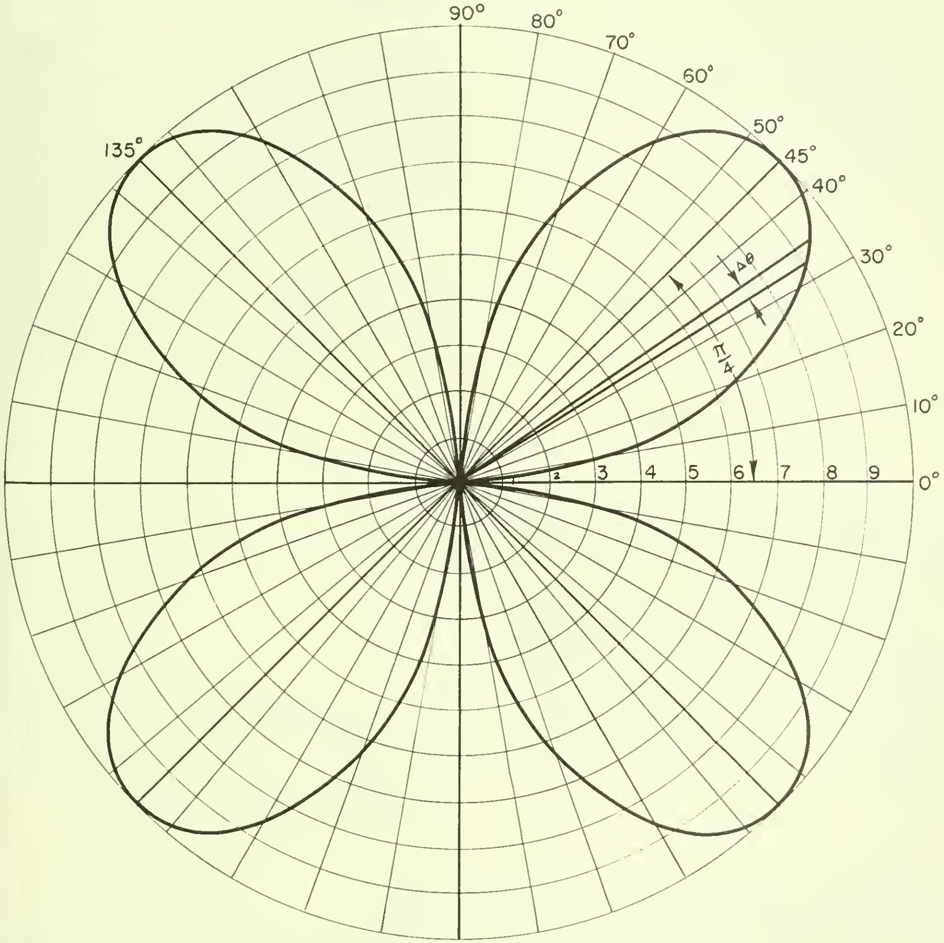

Figure 8–23.—$\rho = a \sin 2\theta$.

If the force and the motion are in the same direction, the work is positive. If the force and resulting motion are in opposite directions, the work is negative.

Figure 8–24 shows two positive charges, q_0 and q_1. Charge q_0 is fixed in position, but q_1 is free to move. The charges are alike so that q_0 will tend to push q_1 away along line AB. The work done by q_0 on q_1 will be positive for both the force and motion are in the same direction. Now let us push against q_1 so as to oppose, but not stop the motion of q_1 away from q_0. The work we do on q_1 under this situation is negative as the force and motion are oppositely directed. If we push harder and stop the motion of

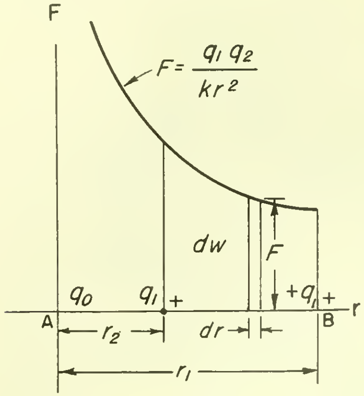

Figure 8–24.—Example 8–13.

q_1, we are doing no work as we have defined work here. There is a force exerted, but no motion results and hence, no distance over which the force acts.

If we exert still greater effort and begin to move q_1 towards q_0, we are doing positive work since both the force and motion are in the same direction. However, the work done by charge q_0 is now negative as the force it exerts is directed oppositely to the motion.

The force one charge exerts on another, separated by a distance r, is given by

$$F=\frac{q_1q_0}{Kr^2}$$

The force is in dynes when q_1 and q_0 are in cgs electrostatic units of charge and r is in centimeters. The constant K, is the permittivity (dielectric) constant. The value of this constant varies with the material, but for air and free space, we may assume it to be unity. For problems in these surroundings, therefore, the force equation reduces to

$$F=\frac{q_1q_0}{r^2}\text{ dynes}$$

We now wish to calculate the work we must do to move a charge in the presence of a second charge. Figure 8–24 illustrates the problem. Both charges are positive and originally separated by a distance r_1.

Example 8–13.

Calculate the expression for the work necessary to move q_1 to a new position r_2, nearer to q_0.

The differential work element is $dw=Fdr$. The force F, is variable so that we must integrate. dr is the differential path element through which the force acts. We form the following integral

$$W=\int dw=-\int_{r_1}^{r_2} Fdr.$$

The negative sign is introduced because the force is directed in the direction of decreasing r. (Recall as we travel along the x axis towards decreasing values of x that we are traveling in a negative direction.)

$$W=-\int_{r_1}^{r_2}\frac{q_1q_0}{r^2}dr=+\frac{q_1q_0}{r}\Big|_{r_1}^{r_2}$$

$$W=\left(\frac{q_1q_0}{r_2}-\frac{q_1q_0}{r_1}\right)\text{ ergs}$$

The work will be a positive quantity for we have specified $r_2 < r_1$.

Example 8–14.

It can be shown that the work, dW, required to move a differential of charge, dq, from one point to another, across which there exists a potential difference (voltage), V, is $dW=Vdq$. What work must be done to charge a capacitor to a voltage V_1?

The charge on a capacitor is given by

$$q=CV$$

and

$$dq=CdV$$

since the capacitance of a given capacitor is constant. From the work relationship provided above we obtain,

$$dW=Vdq=CVdV$$

$$W=\int dW=\int_0^{V_1}CVdV=\frac{1}{2}CV^2\Big|_0^{V_1}$$

$$W=\frac{1}{2}CV_1^2\text{ joules}$$

The units of work in this case will be joules if C is in farads and V in volts.

8–10. Integration Applied to L–R and R–C Series Circuits.

Whenever stored energy within a circuit element is suddenly released by a switching action (with no external source), the response of the circuit can be predicted by the circuit elements. Because of the fact that the response of the circuit to an internal energy (stored) is dependent on the circuit parameters themselves, it is called the "natural response" or the "source free response." A natural response (change of current or voltage) of any R–L or R–C circuit always approaches zero as time tends to infinity for this is due to the energy loss due to the resistor.

We will now derive the natural responses of some very simple circuits such as the R–L and R–C series circuits.

Referring to figure 8–25A, let us suppose that at time $t=0$ the inductor (L) carried an initial current $i=i_0$. We know that:

$$V=V_L+V_r$$

and

$$V=L\frac{di(t)}{dt}+i(t)R.$$

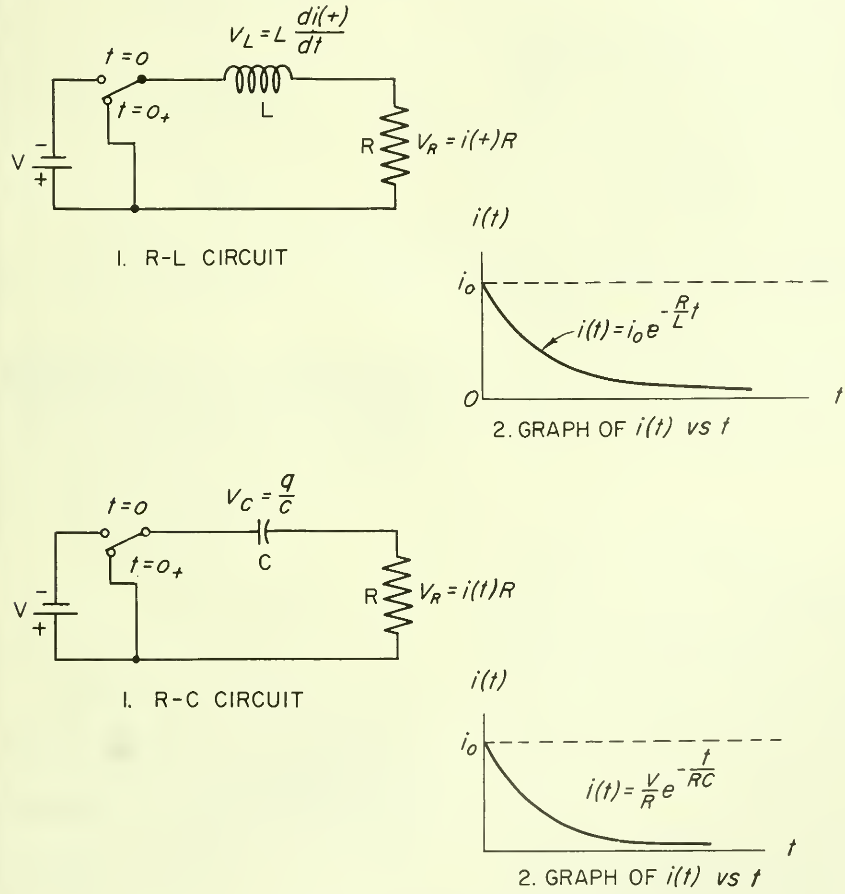

Figure 8–25.—Decay of current in *LR and RC* circuits.

(Note that $i(t)$ is stating the current, i, is a function of time, t.) Now at time $t=0_+$ we know $V=0$ and, $0=L\frac{di(t)}{dt}+i(t)R$ from which

$$\frac{di(t)}{i(t)}=-\frac{R}{L}dt$$

Integrating this expression we have

$$lni(t)+K=-\frac{R}{L}t$$

Now solving for the constant (K) of integration knowing that at $t=0$, $i=i_0$, therefore $lni_0+K=0$ and $K=-lni_0(t)$.
Substituting this value for K in the original expression we have

$$lni(t)-lni_0(t)=-\frac{R}{L}t$$

or

$$ln\frac{i(t)}{i_0(t)}=-\frac{R}{L}t$$

Equate both sides of the expression as powers of the base e:

$$e^{ln\frac{i}{i_0}}=e^{-\frac{Rt}{L}}$$

and remember $e^{ln\frac{i}{i_0}}=\frac{i}{i_0}$

$$\therefore \frac{i}{i_0}=e^{-\frac{Rt}{L}}$$

and

$$i(t)=i_0(t)e^{-\frac{R}{L}t}$$

which is the natural response of the R-L circuit.

What does this current response really look like when we plot its magnitude with respect to time? The graph of the current (fig. 24, A) shows that the initial current in an R-L series circuit will decay to zero, in an exponential form, as time increases to a very large value, if there is no external source applied. This of course is the natural response of such a circuit as there is a resistive element which after a certain time converts all the electrical energy into heat.

Referring now to figure 8–25B we will derive the expression for the natural response of a simple R-C circuit. Assume that the capacitor, at time t=0, carries the initial charge q.

We know that:

$$V=V_c+V_r$$

and

$$V=q/c+i(t)R.$$

Taking the derivative of both sides of the equation we have:

$$\frac{dV}{dt}=\frac{1}{C}\frac{dq}{dt}+\frac{di(t)}{dt}R$$

at time

$$t=0_+, \frac{dV}{dt}=0$$

and therefore

$$-\frac{di(t)}{dt}=\frac{1}{RC}\frac{dq}{dt}$$

also,

$$\frac{dq}{dt}=i$$

thus

$$-\frac{di(t)}{dt}=\frac{i}{RC}$$

and

$$\frac{di}{i}=-\frac{dt}{RC}$$

Integrating the previous equation we have:

$$lni=-\frac{t}{RC}+K.$$

Now $K=lni$ when $t=0$, however, the voltage across the capacitor is zero so that the entire battery voltage appears across the resistor R. Thus, the current i at $t=0$ must be $i_0=\frac{V}{R}$

$$\therefore K=ln\frac{V}{R} \text{ at } t=0.$$

then

$$lni(t)=-\frac{t}{RC}+ln\frac{V}{R}$$

$$e^{lni(t)}=e^{-\frac{t}{RC}}\cdot e^{ln\frac{V}{R}}$$

AND

$$i(t)=\frac{V}{R}e^{-\frac{t}{RC}}$$

which is the natural response of this circuit.

Note that the graph in figure 8–25B verifies that the current will decay in an exponential form as time increases to a large value.

8–11. Approximate Integration by the Trapezoidal Rule.

In some cases it is impossible to express the indefinite integral $\int f(x)dx$ in terms of the basic functions. But in many cases only the approximate value of the definite integral $\int_a^b f(x)dx$ is needed. These cases arise frequently in engineering problems where values are obtained from experimental data.

Suppose this definite integral, $\int_a^b f(x)dx$, is considered to be the area under the curve of $f(x)$ from $x=a$ to $x=b$ (fig. 8–26). Also, consider the interval $b-a$ is divided into n equal parts each of width $\frac{b-a}{n}$.

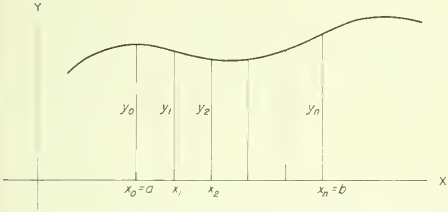

Figure 8–26.—Approximate integration by the trapezoidal rule.

Then the sum of the trapezoidal areas formed under the curve will be an approximation of the area under the curve from $x_0=a$ to $x_n=b$. Saying that $\Delta x=\frac{b-a}{n}$, then the sum of the areas is equal to

$$\frac{1}{2}(y_0+y_1)\Delta x+\frac{1}{2}(y_1+y_2)\Delta x+\ .\ .\ .\ .\ .$$
$$+\frac{1}{2}(y_n-1+y_n)\Delta x.$$

Therefore, the trapezoidal rule is:

$$\int_{x_0}^{x_n} f(x)dx=\frac{\Delta x}{2}(y_0+2y_1+2y_2+\ .\ .\ .\ .\ .$$
$$+2y_{n-1}+y_n)$$

8–12. Approximate Integration by Simpson's Rule

An approximate solution to a definite integral may also be found using Simpson's Rule. Using the same notation as in figure 8–26 and Δx equaling $\frac{b-a}{n}$, Simpson's Rule is:

$$\int_a^b f(x)dx=\frac{\Delta x}{3}\,[y_0+4(y_1+y_3+y_5+\ .\ .\ .)$$
$$+2(y_2+y_4+y_6+\ .\ .\ .)+y_n],$$

giving the symbol y_0 the value of the first ordinate, y_n the last ordinate, and n being an even number.

This rule is based upon the use of parabolic arcs to approximate the curve between two ordinates. The trapezoidal rule was based upon straight lines. Thus, Simpson's Rule gives a better approximation of the area under the curve than the Trapezoidal Rule, where the number of intervals n is about the same.

8–13. Mechanical Integration

Another method of approximating an integral is by mechanical means. Once the curve of the function to be integrated is plotted, the area under the curve may be obtained by use of the planimeter. This is a mechanical device by means of which you can compute the area of an irregular figure after tracing the outline of the figure with a tracing point attached to the mechanism. Figure 8–27 shows the operational principle of the so-called polar planimeter, the type most commonly used.

The principal parts of the instrument are two bars, PA and AB. Bar PA swings around a fixed point called the pole of the instrument. The two bars are freely joined to each other at point A. Bar AB travels over the paper on a wheel R, which revolves

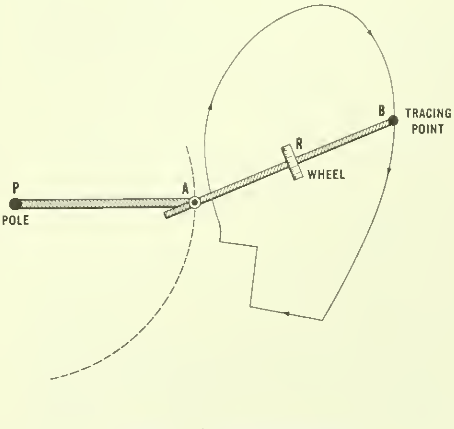

Figure 8–27.—The polar planimeter.

as the tracing point at B moves along the outline of the figure. As the wheel revolves, the revolutions it makes are recorded by a dial on the instrument. The dial is graduated for direct reading to the nearest 0.01 revolution, and is equipped with a vernier for reading to the nearest 0.001 revolution.

The instrument can be set for any desired scale from about $1'' = 20'$ to about $1'' = 200'$. In using the instrument, you first determine the planimeter constant, meaning the area represented by a single revolution of the wheel. The area of the figure is equal to the product of the planimeter constant multiplied by the number of revolutions the wheel makes as you trace the entire outline of the figure.

CHAPTER 9

SERIES

9–1. Definitions

Before discussing series we first must define certain terms used to describe series.

A succession of numbers which follow each other in a definite order is called a SEQUENCE. The successive numbers are called TERMS of the sequence. Each of the following are sequences.

(a) 2, 4, 6, 8, 10, 12, ...

(b) 10, 7, 4, 1, −2, −5, ...

(c) 2, 4, 8, 16, 32, 64, 128, ...

(d) $9, -3, 1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots\ldots$

In the preceding example (a), each term except the first is formed by adding 2 to the preceding term. In example (d), each term except the first is formed by multiplying the preceding term by $-\frac{1}{3}$.

A sequence of numbers in which any term after the first may be obtained from the preceding term by adding to it a fixed number, called the COMMON DIFFERENCE, is called an ARITHMETIC PROGRESSION. Example (a) in the preceding paragraph is an arithmetic progression.

A sequence of numbers in which any term after the first may be obtained from the preceding term by multiplying it by a fixed number, called the COMMON RATIO, is called a GEOMETRIC PROGRESSION. Example (c) from above is a geometric progression.

Let $u_1, u_2, u_3, \ldots\ldots u_n, \ldots\ldots$ be any unending sequence of real numbers positive or negative. The expression $u_1 + u_2 + \ldots . + u_n + \ldots$, is called an INFINITE SERIES when the terms are formed according to some law of succession.

9–2. Properties of Series

In the series $u_1 + u_2 + u_3 \ldots . + u_n + \ldots .$, let S_n represent the sum of the first n terms; that is, let

$$S_1 = u_1$$

$$S_2 = u_1 + u_2$$

$$S_3 = u_1 + u_2 + u_3$$

$$S_4 = u_1 + u_2 + u_3 + u_4$$

$$S_n = u_1 + u_2 + u_3 + \ldots . + u_n.$$

Example 9–1.

In the geometric series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots\ldots$ find S_1, S_3 and $S_n = \frac{u_1[1-r^n]}{1-r}$ where u_1 is the first term, r is a common ration $\frac{u_{n+1}}{u_n}$, and n is the number of terms. (Proof of $S_n = \frac{u_1(1-r^n)}{1-r}$: A geometric series has the general form of $\sum_{k=1}^{\infty} u_1 r^{k-1} = u_1 + u_1 r + u_1 r^2 + \ldots . + u_1 r^{n-1} + \ldots .$

From the identity

$1 - r^n = (1-r)(1 + r + r^2 + \ldots . + r^{n-1})$ the sum, S_n, of a geometric series is

$$S_n = u_1 \sum_{k=1}^{n} r^{k-1} = \frac{u_1(1-r^n)}{1-r}.\Big)$$

$$S_1 = 1$$

$$S_3 = 1 + \frac{1}{3} + \frac{1}{9} = \frac{13}{9}$$

$$S_n = 1 + \frac{1}{3} + \frac{1}{9} + \ldots . + \frac{1}{3^{n-1}} = \frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}$$

Example 9–2.

In the arithmetic series $1 + 3 + 5 + 7 + \ldots\ldots$ find S_1, S_3 and $S_n = \frac{n}{2}(u_1 + u_n)$ where n is the number of terms, u_1 is the first term, and u_n the nth term.

(Proof of $S_n = \frac{n}{2}(u_1 + u_n)$: The sum of the first n terms of an arithmetic series may be written in two forms of

$$S_n = u_1 + (u_1 + d) + (u_1 + 2d) + \dots + [u_1 + (n-1)d]$$

$$S_n = u_n + (u_n - d) + (u_n - 2d) + \dots + [u_n - (n-1)d]$$

where d is the common difference.
By addition of the two forms we have

$$2S_n = (u_1 + u_n) + (u_1 + u_n) + (u_1 + u_n) + \dots + (u_1 + u_n)$$

or

$$2S_n = n(u_1 + u_n)$$

Thus,

$$S_n = \frac{n}{2}(u_1 + u_n).)$$

$$S_1 = 1$$

$$S_3 = 1 + 3 + 5 = 9$$

$$S_n = 1 + 3 + 5 + 7 + \dots + (2n-1) = n^2$$

Example 9–3.

In the alternating series,

$1 - 1 + 1 - 1 + 1 - \dots$, find S_1, S_3, S_4, and S_n

$S_1 = 1$
$S_3 = 1$
$S_4 = 0$
$S_n = 1$ or 0 respectively as n is odd or even.

The above examples illustrate the three cases which may occur in an infinite series. They are as follows:

Case 1: S_n approaches a limit, S, as n increases without limit (Example 9–1).

Case 2: S_n attains a larger value than any assigned value of n greater than 1 (Example 9–2).

Case 3: S_n remains finite and does not approach a limit as n increases without limit (Example 9–3). Zero is a finite number.

The series that fall into Case 1, are called CONVERGENT series. Those which fall into Cases 2 and 3 are called DIVERGENT series.

A preliminary test for divergence which may save considerable time is to examine the nth term, u_n, as n approaches infinity. If u_n is not zero the series is divergent. If u_n is zero the series may be either divergent or convergent so that further tests are required.

If the sum, S_n, of the series approaches a limit as n approaches infinity the series is convergent. If it does not approach a limit the series is divergent.

The preliminary test for divergence may be written as

$$\lim_{n \to \infty} u_n \neq 0$$

Thus it may be repeated: If the limit (as n approaches infinity) of u_n is not zero the series is divergent.

Example 9–4.

Is the infinite series $1 + 2 + 3 + 4 + \dots$ convergent or divergent?

$$u_n = \frac{n}{2}(1+n)$$

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{n}{2}(1+n) = \infty \neq 0$$

thus the series is divergent.

Example 9–5.

Is the series $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$ convergent or divergent?

$$u_n = \frac{1}{n}$$

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = 0$$

thus the series may or may not be convergent. At this time the answer can not be obtained.

The convergence of a series does not change when a finite number of terms is omitted because the sum of the finite terms omitted and the sum of the depleted series is equal to the sum of the original series.

For example in the geometric series $27 + 9 + 3 + 1 + \frac{1}{3} + \frac{1}{9} \dots$ the sum S_n of the series is

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{u_1(1-r^n)}{1-r} = \lim_{n \to \infty} \frac{27\left(1 - \frac{1}{3^n}\right)}{1 - \frac{1}{3}}$$

$$= \lim_{n \to \infty} \frac{27\left(1 - \frac{1}{3^\infty}\right)}{1 - \frac{1}{3}} = \frac{27(1-0)}{\frac{2}{3}} = 40\,\frac{1}{2}$$

where $u_1 = 27$ and $r = \frac{9}{27} = \frac{1}{3}$ ·

Thus the given series converges on $40\frac{1}{2}$.

If the first three terms of the series are removed, the depleted series $1+\frac{1}{3}+\frac{1}{9}\ .\ .\ .\ .$ will converge on

$$S'_n = \lim_{n\to\infty} \frac{u_1(1-r^n)}{1-r} = \lim_{n\to\infty} \frac{1\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}}$$

$$= \lim_{n\to\infty} \frac{1\left(1-\frac{1}{3^\infty}\right)}{1-\frac{1}{3}} = \frac{1-0}{\frac{2}{3}} = 1\frac{1}{2} \text{ where } u_1 = 1$$

and $r=\frac{1}{3}$.

Thus the depleted series converges on $1\frac{1}{2}$.

The sum of the terms omitted $(27+9+3=39)$ is a finite number which added to the sum $(S'_n=1.5)$ of the depleted series is equal to $39+1\frac{1}{2}$ or $40\frac{1}{2}$ which is the sum, S_n, on which the original series converged.

We may assume that if S_n of an infinite series of positive terms is always less than some definite number, the series is convergent. With this assumption we can discuss comparison tests for convergence and divergence.

Given a series of positive terms $u_1+u_2+\ .\ .\ .\ .\ +u_n+\ .\ .\ .\ .$, to be tested for convergence. Suppose we find a series of positive terms $t_1+t_2\ .\ .\ .\ +t_n+\ .\ .\ .$ which is known to be convergent.

If $u_n \leqq t_n$ for all corresponding terms, then the sum of the u series is equal to or less than the sum of the convergent t series, and the u series is convergent.

Given a series of positive terms $u_1+u_2+\ .\ .\ .\ .\ +u_n+\ .\ .\ .\ .$, to be tested for divergence. Suppose we find a series of positive terms $t_1+t_2\ .\ .\ .\ .\ +t_n+\ .\ .\ .\ .$ which is known to be divergent. If $u_n \leqq t_n$ for all corresponding terms, then the u-series is divergent.

Convergent Series for Comparison

The following series are convergent:

1. $1+\frac{1}{2^a}+\frac{1}{3^a}+\ .\ .\ .+\frac{1}{n^a}+\ .\ .\ .\ (a>1)$
2. $1+\frac{1}{2^2}+\frac{1}{3^3}+\ .\ .\ .+\frac{1}{n^n}+\ .\ .\ .\ .$
3. $a+ar+ar^2+\ .\ .\ .+ar^{n-1}+\ .\ .\ .\ (r<1)$
4. $\frac{1}{1(2)}+\frac{1}{2(3)}+\frac{1}{3(4)}+\ .\ .\ .\ \frac{1}{n(n+1)}\ .\ .\ .$

Divergent Series for Comparison

The following series are divergent:

1. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ .\ .\ .+\frac{1}{n}+\ .\ .\ .$
2. $a+ar+ar^2+\ .\ .\ .\ .\ ar^{n-1}+\ .\ .\ .\ .\ (r\geqq 1)$

Another test which can readily be used on series whose terms are not all of the same size is the ratio test.

Consider the ratio, $\frac{u_n+1}{u_n}$, of the $(n+1)^{th}$ to the nth term. Suppose that this ratio approaches a limit as n approaches ∞. We may now state the ratio test as follows:
(The parallel enclosure lines indicate absolute (numerical) value without regard to sign.)

1. If $\lim_{n\to\infty}\left|\frac{u_n+1}{u_n}\right| < 1$, the series converges
2. If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1$, the series diverges
3. If $\lim_{n\to\infty}\left|\frac{u_n+1}{u_n}\right| = 1$, the test fails.

Example 9–6.

A. Test for convergence and divergence of the series:

$$1+\frac{1}{3}+\frac{1}{9}\ .\ .\ .\ .\ .$$

$$u_n = \frac{1}{3^{n-1}} = 3^{1-n}$$

$$u_{n+1} = \frac{1}{3^n} = 3^{-n}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{3^{-n}}{3^{1-n}}\right|$$

$$= \lim_{n\to\infty} 3^{-n-(1-n)} = \frac{1}{3}$$

$\frac{1}{3}<1$ and the ratio test (condition (1) above) indicates the series to be convergent.

It has been shown previously this series is convergent on the finite number $1\frac{1}{2}$ by the method

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{u_1(1-r^n)}{1-r}$$

$u_1 = 1$

$r = \frac{1}{3}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}} = \frac{1(1-0)}{\frac{2}{3}} = 1\frac{1}{2}$$

B. Apply the ratio test for convergence and divergence of the series

$$1 - 2x + 3x^2 - 4x^3 + \ldots.$$

The ratio of terms, u_{n+1} to u_n is found as follows:

$$|u_{n+1}| = |(n+1)x^n|$$
$$|u_n| = |nx^{n-1}|$$
$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)x^n}{nx^{n-1}}\right| = \left|\frac{n+1}{n}x\right|$$

The limit of the ratio of u_{n+1} to u_n is,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}x\right|$$
$$= \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)x\right|$$
$$= |x|$$

Applying condition (1) of the ratio tests, this series is convergent if x lies between $+1$ and -1. Applying condition (2), the series is divergent if $|x| > 1$.

9–3. Power Series

The series in the form of $a_0 + a_1x + a_2x^2 + \ldots. + a_nx^n + \ldots.$, having positive integral powers of the variable x and constant coefficients a_i, is called a power series of x.

The variable x may be assigned a value and the series becomes a series of constants. This series may or may not converge depending on the value assigned to x.

Omitting the first term a_0, let us apply the ratio test to the power series to determine for what values of x the series converges.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_nx^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\left|x\right| = R$$

If R is less than unity ($R < 1$) the series converges. This says the power series will converge if

$$\left|x\right|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

or, transposing $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ to the other side of the inequality,

$$\left|x\right| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

Thus we can say the power series converges when: $\left|x\right| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ and diverges when $\left|x\right| > \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$. The series may or may not converge when $\left|x\right| = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$.

The entire range of values of x, for which the series converges makes up the interval of convergence, the end points of the interval being $x = \pm\lim_{n\to\infty}\left|\frac{a_n}{a_n+1}\right|$.

Example 9–7.

Find the interval of convergence of the following series.

$$\frac{x}{2} + \frac{2^2x^2}{2^2} + \frac{3^2x^3}{2^3} + \ldots.$$

$$u_n = \frac{n^2x^n}{2^n} \text{ and } u_n + 1 = \frac{(n+1)^2x^{n+1}}{2^{n+1}}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2x}{2n^2}\right|$$
$$= \lim_{n\to\infty}\left|\left(\frac{n^2}{2n^2} + \frac{2n}{2n^2} + \frac{1}{2n^2}\right)x\right| = \left|\frac{x}{2}\right|$$

Thus the series is convergent for $\left|\frac{x}{2}\right| < 1$ or $\left|x\right| < 2$ or $-2 < x < 2$.

Exercise 9–1.

Find the interval of convergence for the following series.

1. $1+\left(\frac{x}{2}\right)+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+.\ .\ .\ .\ .$

2. $x+\frac{x^2}{2}+\frac{x^3}{3}+.\ .\ .\ .\ .$

$$\left(\text{Note: } \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \lim_{n\to\infty}\left|\frac{1}{\frac{n+1}{n}}\right| = \lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right| = \frac{1}{1+\frac{1}{\infty}} = \frac{1}{1+0} = 1\right)$$

3. $1+\frac{2x}{3}+\frac{3x^2}{3}+\frac{4x^3}{3}+.\ .\ .\ .$

9–4. MacLaurin's Series

Given a function $f(x)$ which is written in the form of an infinite series, the function is said to be expanded in an infinite series and the infinite series is said to represent the function in the interval of convergence. One such expansion is called a MacLaurin's Series and will be described later in this section.

Example 9–8.

The geometric series,

$1+x+x^2+.\ .\ .\ .+x^n+.\ .\ .\ .$ converges to $\frac{1}{1-x}$ for $|x|<1$.

(Note that $S_n=\frac{u_1(1-r^n)}{1-r}$ from example 9–1.

$$S_n=\frac{u_1(1-r^n)}{1-r}=\frac{u_1}{1-r}-\frac{u_1r^n}{1-r}$$

Now then the $\lim_{n\to\infty} S_n=\frac{u_1}{1-r}$ and the preceding geometric series converges to $\frac{1}{1-x}$.) Therefore $\frac{1}{1+x}=1+x+x^2+.\ .\ .\ .+x^n+.\ .\ .\ .,\ -1\leq x<1$ and the function $\frac{1}{1-x}$ is expanded in an infinite series. The series represents the function only when $-1\leq x<1$ for the series converges with these limits on x, and diverges for all other values. The function exists for all values of x except when $x=1$.

Let us now find the infinite series that represents the sin x, and assume that it will be in the form of the power series.

$$f(x)=A+Bx+Cx^2+Dx^3+.\ .\ .\ .$$

In this case $f(x)=\sin x$. Our problem now is to determine the value of each coefficient, A, B, C, and so forth.

First, let the power series exist for the sine function. Then differentiate with respect to x several times:

1. $f(x)=\sin x=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+.\ .\ .\ .\ .$
2. $f'(x)=\cos x=B+2Cx+3Dx^2+4Ex^3+5Fx^4+.\ .\ .$
3. $f''(x)=-\sin x=2C+6Dx+12Ex^2+20Fx^3+.\ .\ .\ .$
4. $f'''(x)=-\cos x=6D+24Ex+60Fx^2+.\ .\ .$
5. $f''''(x)=\sin x=24E+120Fx+.\ .\ .\ .$
6. $f'''''(x)=\cos x=120F+.\ .\ .\ .$

Assuming that the above equations are true when $x=0$, we find the coefficients A, B, C, and so forth as follows:

1. $f(0)=\sin 0=A$ or $A=0$
2. $f'(0)=\cos 0=B$ or $B=1$
3. $f''(0)=-\sin 0=2C$ or $C=0$
4. $f'''(0)=-\cos 0=6D$ or $D=-\frac{1}{6}$
5. $f''''(0)=\sin 0=24E$ or $E=0$
6. $f'''''(0)=\cos 0=120F$ or $F=\frac{1}{120}$

Now we have evaluated the needed coefficients. Substituting these results into (1),

$$\sin x=x-\frac{x^3}{6}+\frac{x^5}{120}-.\ .\ .\ .\ .+\frac{(-1)^{(n+1)}x^{(2n-1)}}{(2n-1)!}$$

This series enables us to calculate the sine of an angle by substituting the value of the angle, in radians, into the series.

The procedures which are followed to arrive at the required series are the same procedures that the MacLaurin's Series represents in its general form to expand a function.

The general form of the MacLaurin's Series is as follows:

$$f(x)=f(0)+f'(0)x+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}$$
$$+\frac{f''''(0)x^4}{4!}+\ldots.$$

This expansion theorem may be used when we know the value of the function when $x=0$ and when we can calculate all the successive derivatives at the point where $x=0$. If $f(0)$ or any of its derivatives, $f^n(0)$, are undefined, the MacLaurin Series can not exist.

Example 9–8.

Find the expansion of

$$\int_0^x \frac{\sin x\,dx}{x}.$$

We have already found that $\sin x = x-\frac{x^3}{6}+\frac{x^5}{120}$ $\ldots+$ and we note $6=3!$, $120=5!$ and so forth. Thus the exponent of x is equal to factorial r where $r=1\cdot2\cdot3\ldots r$ and the expression becomes,

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots.$$

Dividing both sides of the equation by x,

$$\frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}$$

and, from chapter 7,

$$\int_0^x \frac{\sin x\,dx}{x}=\int_0^x dx-\int_0^x \frac{x^2dx}{3!}+\int_0^x\frac{x^4dx}{5!}-\int_0^x\frac{x^6dx}{7!}$$
$$=x-\frac{x^3}{3\cdot3!}+\frac{x^5}{5\cdot5!}-\frac{x^7}{7\cdot7!}+\ldots.$$

Exercise 9–2.

Using MacLaurin's Series, find the expansion of the following functions:

1. $f(x)=e^x$
2. $f(x)=\tan x$

9–5. Taylor's Series

The Taylor Series is very similar to MacLaurin's Series for both are power series. A function $f(x)$ may be expanded about a point "a" instead of about the origin. That is, $f(x)$ can be represented by the series:

$$f(x)=a_0+a_1(x-a)+a_2(x-a)^2+\ldots.$$
$$+a_n(x-a)^n+\ldots.$$

The coefficients a_1 are computed by repeated differentiations of the above relation and evaluations at the point $x=a$.

The general form of Taylor's Series is as follows:

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2$$
$$+\ldots.\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots.$$

When $a=0$, we have the MacLaurin's Series. Taylor's Series may be used to expand functions when we have the value $f(a)$ of the function for some nearby value a of the independent variable and the values $f'(a)$, $f''(a)$, etc.

Example 9–9.

Find the expansion by Taylor's Series of $\sin x$ about the point a.

$f(x)=\sin x$
$f'(x)=\cos x$
$f''(x)=-\sin x$
$f'''(x)=-\cos x$ etc.

Substitute these values into the general form of Taylor's Series to obtain:

$$\sin x=\sin a+\cos a\,(x-a)$$
$$-\frac{\sin a\,(x-a)^2}{2!}-\frac{\cos a\,(x-a)^3}{3!}+\ldots.$$

The Taylor Series may possess certain advantages over the MacLaurin Series.

1. It may exist for functions having no MacLaurin's Series.
2. The Taylor Series may converge more rapidly for certain values of "a" in the general expression than does the MacLaurin's Series for the same function.

Exercise 9–3.

Find the Taylor's Series of the following functions:

1. $f(x)=l\,nx$, in powers of $x-a$.
2. $f(x)=\cos x$, in terms of $x-a$.

Notice that one may easily compute the values needed to make tables of the trigonometric, logarithmic, and exponential functions by the use of series.

CHAPTER 10

INTRODUCTION TO DIFFERENTIAL EQUATIONS

10–1. Definitions

Before one can study differential equations, one must be able to recognize them and know their properties. Therefore, we ask, what are differential equations?

A differential equation is an equation containing at least one derivative of a function. Some examples are:

$$\frac{dy}{dx}=\tan x \tag{1}$$

$$\left(\frac{d^2y}{dx^2}\right)^4+Cy=0 \tag{2}$$

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=E\omega\cos\omega t \tag{3}$$

$$\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial y^2}=0 \tag{4}$$

$$\frac{d^3x}{dy^3}+x\left(\frac{dx}{dy}\right)^6-3xy=0 \tag{5}$$

Equations (1), (2), (3), and (5) contain ordinary derivatives and are called ORDINARY DIFFERENTIAL EQUATIONS. Equation (4) contains partial derivatives and is called a PARTIAL DIFFERENTIAL EQUATION. In this chapter we will only concern ourselves with ordinary differential equations.

The ORDER of a differential equation is the order of the highest ordered derivative contained in the equation. Example equations (2) and (3) are of the "order two." Equation (5) is of the "order three."

The DEGREE of an ordinary differential equation is its algebraic degree in the highest ordered derivative present in the equation. Example equation (1) is of "degree one" and equation (2) is of "degree four." Equation (5) is of "degree one."

A solution of a differential equation is any relation, free from derivatives, which involves one or more of the variables and which is consistent with the differential equation. For example, $y=e^{2x}$ is a solution of the equation

$$\frac{dy^2}{dx^2}+\frac{dy}{dx}-6y=0.$$

This can be shown to be true by direct substitution.

$$\frac{dy^2}{dx^2}+\frac{dy}{dx}-6y=4e^{2x}+2e^{2x}-6e^{2x}=0$$

10–2. Solutions of a Different Equation

The solutions of differential equations appear in certain forms. One form is termed a GENERAL SOLUTION. For example the differential equation $\frac{dy}{dx}=y$ has a solution of $y=Ce^x$ where C is an arbitrary constant. This solution is a general solution for it contains an arbitrary constant. Notice that this differential equation is a first-order equation and its solution contains one constant (C). A second-order equation will have a general solution containing two arbitrary constants, etc.

When specific values are assigned to any of the arbitrary constants in a general solution, we then refer to the solution as a PARTICULAR SOLUTION. The values of the arbitrary constants may often be established by known BOUNDARY CONDITIONS. In physical problems the conditions are generally INITIAL VALUES of the quantities involved in the differential equation. (Refer to chapter 7, section 6 for further information.)

We now will arrive at a differential equation by eliminating the arbitrary constants in the general solution. In a sense we start with the answer and find the problem. By doing this it will give us a feeling for the kinds of solutions to be expected.

Example 10–1.

Eliminate the arbitrary constants C_1 and C_2 from the relation (general solution).

$$y = C_1e^x + C_2e^{-5x} \qquad (6)$$

Since two constants are to be eliminated, we differentiate the general solution, equation (6), twice with respect to x to obtain

$$y' = C_1e^x + (-5)C_2e^{-5x} \qquad (7)$$

$$y'' = C_1e^x + 25C_2e^{-5x} \qquad (8)$$

$$\left(y' = \frac{dy}{dx} \text{ and } y'' = \frac{dy^2}{dx^2}\right)$$

The elimination of C_1 from equations (7) and (8) yields

$$y'' - y' = 30C_2e^{-5x};$$

the elimination of C_1 from equations (6) and (7) yields

$$y' - y = -6C_2e^{-5x}$$

Therefore,

$$y'' - y' = -5(y' - y)$$

and

$$y'' + 4y' - 5y = 0 \qquad (9)$$

Notice that the above differential equation (9) is of the second order and that the general solution contains two arbitrary constants C_1 and C_2.

10–3. Differential Equations of the First Order and Degree

We now will take up the solving of differential equations. Our work will be limited to a few important kinds of equations.

The general equation for differential equations of the first order and degree is

$$Mdx + Ndy = 0 \qquad (10)$$

where M and N may be functions of both x and y. A more limited form of (10) above is when M is a function of x alone or a constant and N is a function of y alone or a constant. In this form the VARIABLES CAN BE SEPARATED and integration performed on them to produce the solution. The following example will illustrate the separation of variables.

Example 10–2.

Solve the equation

$$2(y + 3)dx - xy\,dy = 0.$$

Transpose and divide by $x(y+3)$ to separate the variables. This gives:

$$\frac{2dx}{x} - \frac{ydy}{y+3} = 0$$

or

$$\frac{2dx}{x} - \left(1 - \frac{3}{y+3}\right)dy = 0$$

Now, by direct integration we obtain the solution.

$$2\int\frac{dx}{x} - \int dy + 3\int\frac{dy}{y+3} = 0$$

(Using rules (5) and (1) section 7–8).

$$2lnx - y + 3\,ln(y+3) = C \qquad (11)$$

The above solution is correct, but the presence of two logarithmic terms suggests that we put the arbitrary constant in logarithmic form and simplify. Thus directly from (11) we may write the solution as

$$2ln\,x - y + e\,ln(y+3) + ln\,C_1 = 0$$

where C_1 is a different arbitrary constant from C. Further simplification yields

$$y = 2\,lnx + 3\,ln(y+3) + ln\,C_1,$$

and thus the final solution is

$$e^y = C_1x^2\,(y+3)^3.$$

Exercise 10–1.

Obtain a general solution of:

1. $my\,dx = nx\,dy$
2. $\sin x \sin y\,dx + \cos x \cos y\,dy = 0$
3. $(1-x)\,y' = y^2$

If in a differential equation, the dependent variable and its derivatives appear in no powers other than the first, the equation is said to be LINEAR. A linear equation of the first order has the standard form of

$$dy + P(x)\,y\,dx = Q(x)dx.$$

($P(x)$ and $Q(x)$ are functions of the variable x.)

One can solve this type of equation by finding an INTEGRATING FACTOR S, must be found so that when $dS = PS\,dx$ and the factor is applied to the general form above, we obtain $S\,dy + y\,PS\,dx = SQ\,dx$, from which the general form will reduce to

$$S dy + y\, dS = SQ\, dx \qquad (12)$$

We recall from our work in differentiation that the differential of a product is $d(uv) = v\,du + u\,dv$. thus (12) reduces to

$$d(Sy) = SQ\,dx. \qquad (13)$$

Integrating both sides of the equation yields the general solution of (13) as

$$Sy = \int SQ\,dx + C \qquad (14)$$

which is also the general solution of (12). Therefore the integrating factor S must be found from the condition imposed that

$$dS = PS\ dx.$$

By separating the variables, the equation becomes

$$\frac{dS}{S} = P dx$$

with a general solution of

$$ln\, S = \int P\,dx \text{ or}$$

$$S = e^{\int P\,dx} \qquad (15)$$

Example 10–3.

Solve the equation

$$2(y - 4x^2)dx + x\,dy = 0$$

The equation is linear in y. Dividing by x and putting into general form it becomes

$$2\frac{y}{x}\,dx - 8\,x\,dx + dy = 0$$

$$dy + \frac{2}{x}\,y\,dx = 8\,x\,dx \qquad (16)$$

Note similarity to the standard form:

$$dy + P(x)y\,dx = Q(x)\,dx$$

where $P(x) = \dfrac{2}{x}$ and $Q(x) = 8\,x$

The integrating factor may be found from (15).

$$S = e^{\int P\,dx}$$

$$S = e^{\int \frac{2}{x}dx}$$

$$= e^{2\, ln\, x}$$

$$= (e^{ln\, x})^2 = x^2$$

Multiplying equation (16) through by the integrating factor x^2 we obtain the equation

$$x^2dy + 2xy\,dx = 8x^3dx \qquad (17)$$

Note similarity of (17) to (12):

$$x^2dy + 2\,xy\,dx = 8x^3dx \qquad (17)$$

$$S\,dy + y\,ds = S \cdot Q\,dx \qquad (12)$$

where $S = x^2$, $ds = 2x\,dx$ and $Q = 8x$

The solution of (17) by (14) is then found by substituting x^2 for S and $8x$ for Q in (14):

$$Sy = \int SQ\,dx + C$$

$$x^2y = \int x^2 \cdot 8x\,dx + C$$

$$= 8\int x^3dx + C$$

$$= \frac{8x^4}{4} + C$$

$$x^2y = 2x^4 + C$$

Exercise 10–2.

Find the general solution.

1. $(x^4 + 2y)\,dx - x\,dy = 0$
2. $y' = \csc x - y \cot x$
3. $dy = (x - 3y)$dx

10–4. Homogeneous Linear Differential Equations With Constant Coefficients

The most important type of second-order differential equation is LINEAR, that is, it has the form

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = f(x) \qquad (18)$$

We will now discuss the special case of the above equation when A and B are constants and $f(x)=0$. With these conditions, equation (18) is said to be a second-order, homogeneous, linear differential equation with constant coefficients.

Since $\frac{d(e^{mx})}{dx}=me^{mx}$, it is possible that

$$\frac{d^2y}{dx^2}+A\frac{dy}{dx}+By=0 \tag{19}$$

has a solution of the form e^{mx}. (Note m is a constant such as A and B are constants.) Thus, we try $y=e^{mx}$ as a solution of (19) and obtain

$$m^2e^{mx}+Ame^{mx}+Be^{mx}=0.$$

Since $e^{mx}\neq 0$ we may divide it out and obtain the quadratic equation for m of

$$m^2+Am+B=0. \tag{20}$$

This equation is called the AUXILIARY EQUATION.

Solving (20) for m we obtain:

$$m=\frac{-A\pm\sqrt{A^2-4B}}{2} \tag{20A}$$

If we denote the two roots of (20) by m_1 and m_2, then e^{m_1x} and e^{m_2x} are solutions of equation (19). The general solution of (19) in this case is then

$$y=C_1e^{m_1x}+C_2e^{m_2x}. \tag{21}$$

The reason for this is that if equation (21) is true then the differential equation resulting by elimination of C_1 and C_2 will, in fact, be a second order linear differential equation as shown in equation (19).

In order that (21) be the general solution, m_1 must not equal m_2. But if the two roots of (20) are equal ($m_1=m_2=m$), direct substitution of $y=xe^{mx}$ into (19) shows that it is a solution. For,

if

$$y=xe^{mx}$$

then

$$y'=xme^{mx}+e^{mx}$$

$$y''=xm^2e^{mx}+2me^{mx}$$

and substituting y, y', y'' in the differential equation (19) gives

$$\begin{aligned} y''+Ay'+By &= xm^2e^{mx}+2me^{mx}+Ax\,me^{mx} \\ &\quad +Ae^{mx}+Bxe^{mx} \\ &= e^{mx}[x(m^2+Am+B)+(2m+A)]=0. \end{aligned}$$

The quantity (m^2+Am+B) is equal to zero for m is a root of the auxiliary equation (20) and $2m+A=0$ because $m=\frac{A}{2}$ from equation (20A) when $m_1=m_2=m$ and $\sqrt{A^2-4B}=0$.

Therefore if $m_1=m_2=m$ the general solution of equation (19) is

$$y=C_1e^{mx}+C_2xe^{mx}. \tag{22}$$

Example 10–4.

Find the general solution of $y'-y'-2y=0$. Comparing this equation with (19) $A=-1$ and $B=-2$. Substituting these values in auxiliary equation (20):

$$m^2-m-2=0$$

Solve this quadratic in accordance with (20A):

$$\begin{aligned} m &= \frac{-(-1)\pm\sqrt{(-1)^2-4(-2)}}{2} \\ &= \frac{+1\pm\sqrt{1+8}}{2} \\ &= \frac{1\pm3}{2}=+2 \text{ and } -1 \end{aligned}$$

assume $m_1=+2$, $m_2=-1$ and to obtain the general solution substitute these values in (21) since $m_1\neq m_2$:

$$y=C_1e^{2x}+C_2e^{-x} \text{ Answer.}$$

Example 10–5.

Find the general solution of $y''+6y'+9y=0$. Using equation (20), the auxiliary equation is

$$m^2+6m+9=0$$

and $m_1=-3=m_2=-3=m$. Thus, from equation (22) the general solution is

$$y=C_1e^{-3x}+C_2xe^{-3x} \qquad \text{Answer.}$$

Exercise 10–3.

Find the general solutions of the following:

1. $y'' - 2y' - 3y = 0$
2. $y'' + 4y' + 4y = 0$
3. $y'' = 9y$

10–5. Applications

Due to our limited study of differential equations only a few direct applications will be discussed. Although this chapter is intended to give the student only a limited background in differential equations it is hoped the selected examples will encourage him to seek further study in the subject.

We will now proceed to the discussion of differential equations as applied to electrical circuits. This is best done by giving examples.

From previous study it is known that the current I through a capacitor is equal to $C\,\dfrac{dE}{dt}$ where C is the capacitance and $E(t)$ is the voltage across the capacitor. Similarly, the current through a resistor R is equal to $\dfrac{E(t)}{R}$ and the current through an inductor L is equal to $\dfrac{1}{L}\int E(t)dt$. The algebraic sum of all currents at a junction (node) must be equal to zero. Due to this relation we arrive at a node equation representing this sum. The following examples will show this relation and also the application of differential equations to electrical circuits.

Example 10–6.

What is the equation for the voltage across the resistor at $t > 0$ when the initial voltage across the capacitor is 10 volts and the initial current (I_0) through the inductor is 50 amps (fig. 10–1). We know that

$$I_c(t) = C\,dE(t)/dt$$

$$I_L(t) = \frac{1}{L}\int E(t)dt$$

At $t = 0$, $I_L(t) = I_0 = -50$ amps.

$$I_R(t) = \frac{E(t)}{R}$$

Thus, we arrive at the node equation:

$$I_c(t) + I_R(t) + I_L(t) = 0$$

$$C\frac{dE(t)}{dt} + \frac{E(t)}{R} + \frac{1}{L}\int E(t)dt = 0 \qquad (23A)$$

Differentiate each term with respect to t:

$$C\frac{d^2E(t)}{dt^2} + \frac{1}{R}\frac{dE(t)}{dt} + \frac{E(t)}{L} = 0 \qquad (23B)$$

Divide by C:

$$\frac{d^2E(t)}{dt^2} + \frac{1}{RC}\frac{dE(t)}{dt} + \frac{E(t)}{LC} = 0 \qquad (23C)$$

This equation is a homogeneous linear differential equation like equation (19), section 10–4, with constant coefficients. Thus, the auxiliary equation is

$$m^2 + \frac{1}{RC}m + \frac{1}{LC} = 0$$

$$m = \frac{-\dfrac{1}{RC} \pm \sqrt{\left(\dfrac{1}{RC}\right)^2 - \dfrac{4}{LC}}}{2}$$

Substituting the given values for R, C, and L into the above equation yields

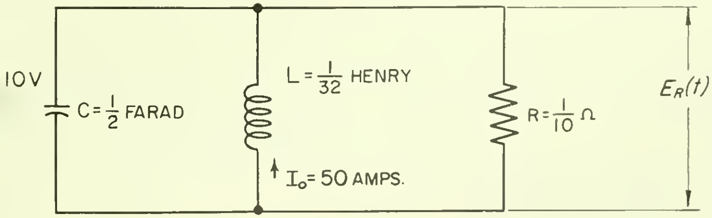

Figure 10–1.—Example 10–6.

$$m=\frac{-\frac{1}{\left(\frac{1}{10}\right)\left(\frac{1}{2}\right)}\pm\sqrt{\left(\frac{1}{\frac{1}{10}\cdot\frac{1}{2}}\right)^2-\frac{4}{\frac{1}{32}\cdot\frac{1}{2}}}}{2}$$

$$=\frac{-20\pm\sqrt{20^2-(4)(64)}}{2}$$

$$=\frac{-20\pm 12}{2}$$

$$=-4 \text{ and } -16$$

$$\therefore m_1=-4$$

$$m_2=-16$$

Thus the general solution for equation (21) is:

$$E(t)=C_1e^{-4t}+C_2e^{-16t} \qquad (24)$$

Solving for C_1 and C_2 we know:

at $t=0$, $E(t)=E_0$ and $I(t)=I_0$ thus,

$$E_0=C_1e^0+C_2e^0$$

$$E_0=C_1+C_2$$

$$\text{and } C_2=E_0-C_1$$

Substitute $C_2=E_0-C_1$ in (24):

$$E(t)=C_1e^{-4t}+(E_0-C_1)e^{-16t} \qquad (25)$$

Substitute (25) in the node equation (23A) which for this example is repeated here as

$$C\frac{dE(t)}{dt}+\frac{E(t)}{R}+\frac{1}{L}\int E(t)dt=0 \qquad (23A)$$

$$C\frac{d[C_1e^{-4t}+(E_0-C_1)e^{-16t}]}{dt}+\frac{C_1e^{-4t}+(E_0-C_1)e^{-16t}}{R}$$

$$+\frac{1}{L}\int [C_1e^{-4t}+(E_0-C_1)e^{-16t}]dt=0$$

Note the second and third terms simplify to $\frac{E_0}{R}$ and I_0 respectively at $t=0$. However it is necessary to differentiate the first term with respect to t in order to solve for C_1. The node equation then becomes:

$$C\frac{d(C_1e^{-4t})}{dt}+C\frac{d(E_0e^{-16t})}{dt}-CC_1\frac{d(e^{-16t})}{dt}$$

$$+\frac{E_0}{R}+I_0=0$$

$$CC_1e^{-4t}(-4)+CE_0e^{-16t}(-16)-CC_1e^{-16t}(-16)$$

$$+\frac{E_0}{R}+I_0=0$$

at $t=0$ the above equation simplifies to:

$$-4CC_1-16CE_0+16CC_1+\frac{E_0}{R}+I_0=0$$

$$12CC_1-16CE_0+\frac{E_0}{R}+I_0=0$$

substituting $C=\frac{1}{2}$, $E_0=10$, $R=\frac{1}{10}$, and $I_0=-50$ in the preceding equation:

$$(12)\left(\frac{1}{2}\right)(C_1)-(16)\left(\frac{1}{2}\right)(10)+\frac{10}{\frac{1}{10}}-50=0$$

$$6C_1-80+100-50=0$$

$$6C_1=30$$

$$C_1=5$$

and then since

$$E_0=C_1+C_2 \qquad \text{and } E_0=10=5+C_2$$

$$C_2=5$$

Thus, the particular solution from equation (24) is:

$$\text{Ans. } E_R(t)=5e^{-4t}+5e^{-16t}$$

Example 10–7.

In figure 10–2A the switch is closed at $t=0$. Prior to this time there was a voltage on C_1 of 10 volts. Find the equation for the voltage $E(t)$ for $t>0$.

Before the switch is closed the total charge is

$$q=CV$$

$$=2(10)$$

$$=20 \text{ coulombs}$$

When the switch is closed at $t=0$ (fig. 10–2B) the voltage across the parallel combination of C_1, C_2, and R is obtained from:

$$q=CV_0$$

$$20=5V_0$$

$$V_0=4 \text{ volts}$$

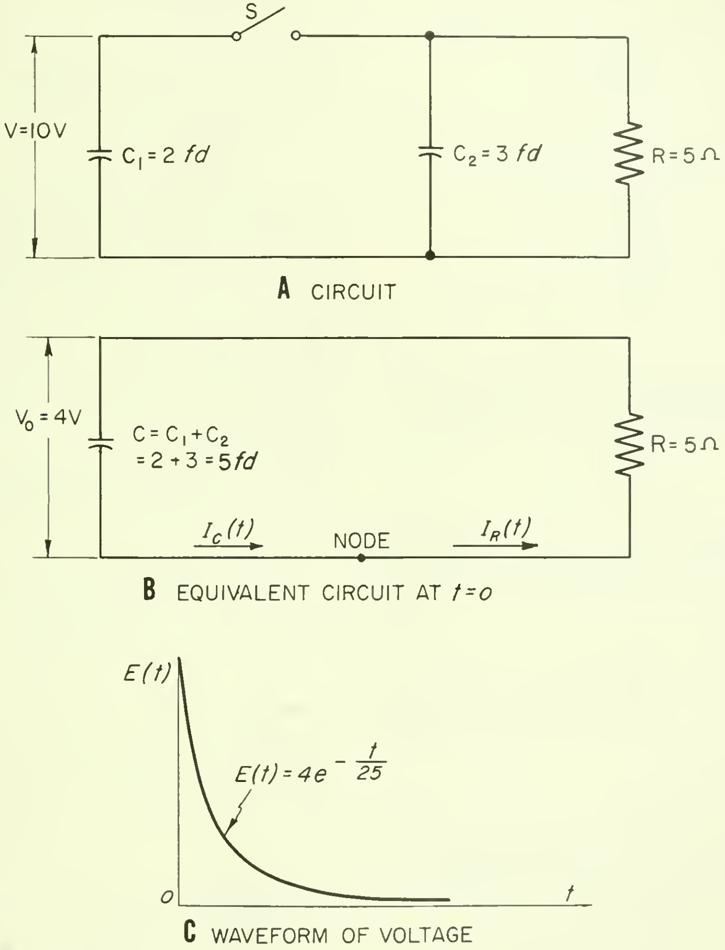

Figure 10-2.—Example 10-7.

Knowing the algebraic sum of the currents at a node must be equal to zero and knowing $I_R(t)=\frac{E(t)}{R}$ and $I_c(t)=C\frac{dE(t)}{dt}$, we have the node equation:

$$\frac{E(t)}{R}+C\frac{dE(t)}{dt}=0$$

Multiply by dt and divide by $CE(t)$ to separate the variables (chapter 10, section 3).

$$\frac{dE(t)}{E(t)}=-\frac{1}{RC}dt \qquad (26)$$

Integrate:

$$\int\frac{dE(t)}{E(t)}=\int-\frac{1}{RC}dt$$

$$lnE(t)=-\frac{t}{RC}+ln\,C_1$$

(C_1 is an arbitrary constant of integration.)

Thus the general solution of equation (26) is

$$E(t)=C_1e^{-\frac{t}{RC}} \qquad (27)$$

Solving for C_1 we know at $t=0$ that $E(t)=E_0=4$ volts. Thus from equation (27) at $t=0$, $C_1=4$ volts. Therefore, the particular solution of equation (26) is:

$E(t)=4e^{-\frac{t}{RC}}$ or from figure 10–2B:

$$E(t)=4e^{-\frac{t}{25}}$$

The graph of the equation is illustrated in fig. 10–2C.

Example 10–8.

Knowing that $i=\frac{dq}{dt}$ where i equals the current and q equals the charge on the capacitor, write the circuit equation for the circuit in figure 10–3.

It is assumed that the student realizes that the sum of the voltages across all elements in the circuit must be equal to zero. Knowing that the voltage across the resistor is $R\frac{dq}{dt}$, across the inductor is $L\frac{d^2q}{dt^2}$, and across the capacitor is $\frac{1}{C}q$ the circuit equation is as follows:

$$E(t)=L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q \text{ Ans.}$$

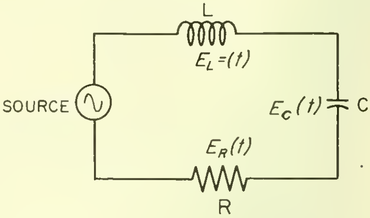

Figure 10–3.—Circuit for example 10–8.

We shall now briefly discuss mechanical systems with reference to differential equations.

Figure 10–4 shows a particle of mass (m) attached to a weightless spring. This mass is subjected to a force which we denote by $F(t)$. The force acts in the x direction causing the mass to move in this direction and the spring to expand. The unstretched spring positions the mass at $x=0$ and the spring force is proportional to the displacement x (Hooks Law) and has a value of x multiplied by "k" which is termed the spring constant.

If there is a frictional force retarding the motion of the mass, it will be proportional to the velocity of the mass. The constant of proportionality will be designated as "f".

Thus, in accordance with Newton's laws, the forcing function may be represented mathematically by the following differential equation:

$$F(t)=m\frac{d^2x}{dt^2}+f\,\frac{dx}{dt}+kx$$

where $\frac{d^2x}{dt^2}$ is equal to the acceleration of mass m.

This equation is similar to one that we have derived before. Compare this equation to the answer given for Example 10–8.

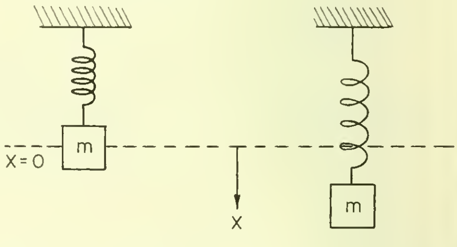

Figure 10–4.—Mechanical system analogous to a series LCR circuit.

Notice that the forcing function, $F(t)$, is similar to $E(t)$. Also notice that the charge (q) on the capacitor is similar to displacement (x), that m corresponds to L, that f corresponds to R, and that k corresponds to $\frac{1}{C}$.

The analogy between mechanical systems and electrical systems has given us a very helpful tool. That tool is the solving of mechanical problems by use of electrical circuits and components.

It is thus evident that certain electrical components put in a prescribed circuit could perform certain mathematical operations including the solving of differential equations. We shall now describe a few of these circuits and components. These circuits and components are used in most analog computers. The mathematical applications will be emphasized.

The basic element of most electronic analog computers is a high-gain d-c amplifier. This element is combined with other elements to perform a specific mathematical operation.

A conventional single line diagram of a high-gain amplifier is given in figure 10–5. The input voltage is represented by e_i and the output voltage is represented by e_o.

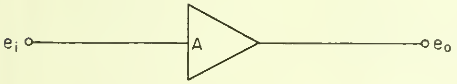

Figure 10–5.—Conventional symbol of high-gain amplifier.

The operation performed by the high-gain amplifier is represented by the equation

$$e_o = -Ae_i$$

where A is a positive constant representing the gain for the amplifier. The minus sign indicates 180° phase shift between e_o and e_i.

When sufficiently large resistances (100k – 10 meg.) are introduced as shown in figure 10–6, the relationship between e_i and e_o is approximated by the expression

$$e_o = -\left(\frac{R_f}{R_i}\right) e_i$$

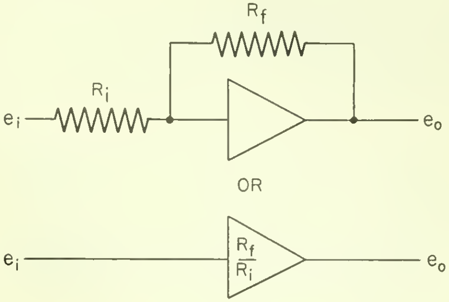

Figure 10–6.—A constant multiplier.

The ratio $\frac{R_f}{R_i}$ is a constant multiplier. When a high resistance resistor and a low-loss capacitor (0.1–2 μfd.) are connected to a high-gain amplifier as shown in figure 10–7, the relationship between e_i and e_o is approximated by the expression.

$$e_o = -\left(\frac{1}{RC}\right)\int_0^t e_i(t)dt + K$$

where K is the initial value of e_o realized by the initial charge on the capacitor. The above expression is an integration multiplied by the constant $\left(\frac{1}{RC}\right)$.

The multiplication of a voltage by a constant less than unity is usually performed by an attenuator consisting of a precision potentiometer. The volt-

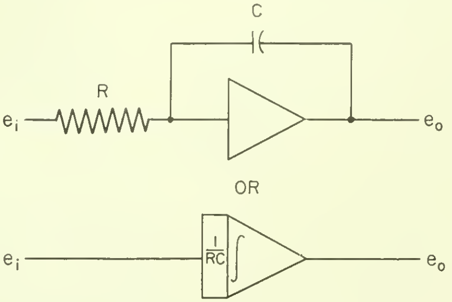

Figure 10–7.—The integrator.

age, e_o, of the potentiometer is set to three-place accuracy by servomechanical or manual means. The circuit diagram and schematic are given in figure 10–8. The relationship between e_i and e_o is

$$e_o = ae_i \text{ where } a < 1.$$

Figure 10–9 shows some of the common relationships between e_i and e_o and their schematic diagrams.

Let us now look at some example problems which will show the student some uses of analog devices in solving problems.

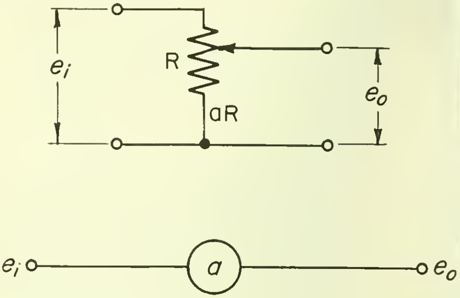

Figure 10–8.—Circuit and schematic of a potentiometer.

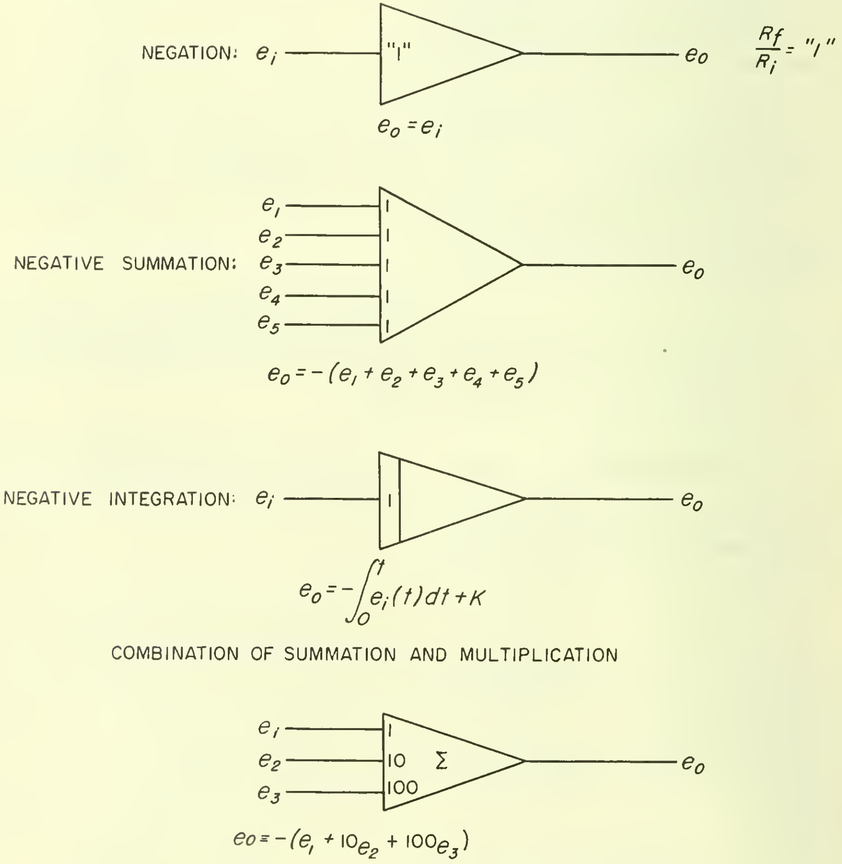

Figure 10–9.—Relation between e_i and e_o.

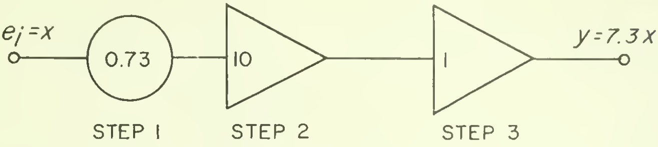

Figure 10–10.—Example 10–9.

Example 10–9.

Show the operational schematic of the expression $y = 7.3x$. Let $x = e_i$ in figure 10–10.

The first step uses a potentiometer to multiply e_i by 0.73 giving an output voltage of $0.73x$.

The second step uses a high-gain amplifier with a constant multiplier of $\frac{R_f}{R_i} = 10$ giving an output voltage of $-7.3x$.

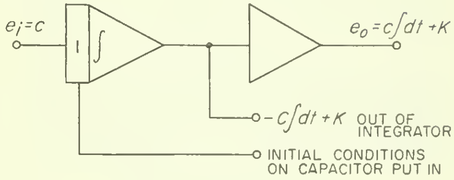

Figure 10–11.—Example 10–10.

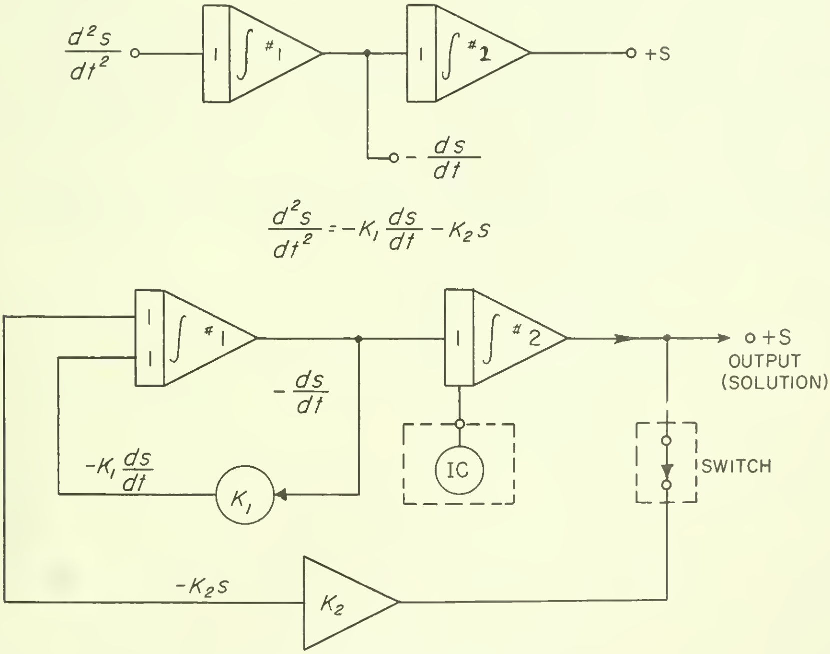

Figure 10–12.—Operation schematic for solving $\frac{d^2s}{dt^2} + K_1\frac{ds}{dt} + K_2 s = 0$.

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The third step uses a high-gain amplifier with a constant multiplier of $\frac{R_f}{R_i} = 1$, giving a negation operation. Thus the output voltage is $7.3x$ which is the desired operation.

Example 10–10.

Solve the first order differential equation $\frac{dx}{dt} = C$ where C is a constant.

The equation may be arranged in this order. $\int dx = C \int dt + K$ where K is the initial condition constant. The schematic is shown in fig. 10–11.

Many physical systems are described by the following differential equation, i.e., they have the same mathematical model (K_1 and K_2 are constants):

$$\frac{d^2s}{dt^2} + K_1 \frac{ds}{dt} + K_2 S = 0 \tag{28}$$

The method of solving this equation is representative of the approach used in electronic analog computers. The approach is a unique one. Note that by rearranging equation (28) we can represent the second derivative term as equal to the negative sum of the other two terms. Why not do this electronically? This concept is illustrated in figure 10–12.

At first glance the circuit shown is puzzling. What is the input voltage? The answer to this question is related to the components shown within the dashed lines. Consider for a moment, the equation that we have mechanized. What does it represent? The equation we have mechanized represents any oscillating system being damped by friction (figure 10–4 for example). When left undisturbed, a mass on a spring, pendulum, or ringing circuit will not do anything. Displace it, strike it, or pulse it and the system will oscillate until it is completely damped out. The components in the dashed boxes represent the elements of a disturbance. The *IC* potentiometer supplies a voltage to the output of the second integrator. Since the output of the second integrator represents displacement, the *IC* voltage represents an "initial condition" (*IC*) corresponding to initial displacement of a mass on a spring, pendulum, or the amplitude of a pulse applied to a ringing circuit. The *IC* voltage charges the feedback capacitor of the second integrator. Closing the start switch applies the disturbance. Electronically it pulses the computer circuits. As a result, the signal passes through a gain of $-K_2$ and is integrated by integrator number 1. A portion of the output signal is fed back through an attenuation of K_1 (we assumed that K_1 is less than unity). Consequently, the circuit starts operating.

Remember that the voltages applied to integrator number 1 are equal to $\frac{d^2s}{dt^2}$. As a result, the two integrators produce the solution to the differential equation noted as the output $+s$ in figure 10–12.

APPENDIX I

MATHEMATICAL SYMBOLS AND ABBREVIATIONS

$+$	Plus
	Positive
$-$	Minus
	Negative
$\pm$	Plus or minus
	Positive or negative
$\mp$	Minus or plus
	Negative or positive
$\times$ or $\cdot$	Multiplied by
$\div$	Divided by
$=$	Equals, as
$\neq$	Does not equal
$\approx$	Equals approximately
$>$	Greater than
$<$	Less than
$\geqq$	Greater than or equal to
$\leqq$	Less than or equal to
$\equiv$	Is identical to
$\rightarrow$ or $\doteq$	Approaches as a limit
$\therefore$	Therefore
$\sqrt{}$	Square root
$\sqrt[n]{}$	nth root
a^n	nth power of a
n!	$1 \cdot 2 \cdot 3 \cdots n$
log	Common logarithm
	Briggsian logarithm

ln or $\log_e$	Natural logarithm
	Hyperbolic logarithm
	Napierian logarithm
e	Base (2.718) of natural system of Logarithms
π	Pi (3.1416)
$\angle$	Angle
$\perp$	Perpendicular to
$\parallel$	Parallel to
( )	Parentheses
[ ]	Brackets
{ }	Braces
a_n	a sub n
sin	Sine
cos	Cosine
tan	Tangent
cot	Cotangent
sec	Secant
csc	Cosecant
$\sin^{-1} a$	Angle whose sine is a: inverse sine a
$P(x,y)$	Rect. coord. of point P
$P(r,\theta)$	Polar coord. of point P
$f(x)$	Function of x
Δy	Increment of y
Σ	Summation of
∞	Infinity
$\int$	Integral of
$\int_a^b$	Integral between the limits of a and b

APPENDIX II

ANSWERS TO EXERCISES

CHAPTER 1

Exercise 1–1

A. 63
B. 12,144
C. 11
D. 1260

Exercise 1–2

a. $x=18,\ y=6$

b. $x=\dfrac{58}{43},\ y=\dfrac{23}{43}$

c. $x=12,\ y=-3$

d. $x=-5,\ y=3,\ z=2$

CHAPTER 5

Exercise 5–1

1. 1
2. $\dfrac{a-1}{3}$
3. ∞
4. $\dfrac{1}{2}$
5. 4
6. 8
7. 0, v_m

Exercise 5–2

1. m
2. $4x$
3. $1-6x$
4. $4a$
5. $-\dfrac{2}{x^3}$
6. $-\dfrac{1}{x^2}$

Exercise 5–3

a. $y'=\dfrac{x^2+2x}{(x+1)^2},\ y''=\dfrac{2}{(x+1)^3}$

b. $y'=-\dfrac{x}{y},\ y''=\dfrac{-r^2}{y^3}$

c. $y'=-\dfrac{2}{(x-1)^2},\ y''=\dfrac{4}{(x-1)^3}$

d. $y'=\dfrac{-y}{2x},\ y''=\dfrac{3y}{4x^2}$

Exercise 5–4

A. 1. (1,2) minimum, $(-1,-2)$ maximum
 2. (2,2) minimum, $(2,-2)$ maximum
 3. 1 ampere, $\dfrac{1}{2}$ volt

B. 1. (a) $t=1.414$ sec.
  (b) $\vec{v}_y=45.25$ ft/sec
 2. (a) $t=2$ sec
  (b) $\leftarrow$
  (c) $\leftarrow$
 3. (6 ft, 9 ft), $\vec{v}_x=2$ ft/sec $\rightarrow$,
  $\vec{v}_y=6$ ft/sec $\uparrow$,
  $\vec{v}=2\sqrt{10}\,\angle 71.6^\circ$, $a_x=0$,
  $\vec{a}_y=2$ ft/sec^2 $\uparrow$,
  $\vec{a}=2$ ft/sec^2

Exercise 5–5

1. 8π in^3/min
2. $\dfrac{3}{2}$ ft/sec
3. $\dfrac{1}{3}$

Exercise 5–6

1. 12%
2. 0.275 in^2

CHAPTER 6

Exercise 6–1

1. $6 \sin^2 2x \cos 2x$
2. $\dfrac{e^{ln x}}{x}$
3. $e^x\left(\dfrac{1}{\sqrt{1-x^2}}+\sin^{-1} x\right)$
4. $ln 10 (\sec^2 x) 10^{\tan x}$
5. $12 \sin^5 2x \cos^6 2x - 10 \sin^7 2x \cos^4 2x$
6. $et^{e-1} + 2e^{2t} + et^2 + \cos t$
7. $3 \tan 3x$
8. $\dfrac{2}{x\dfrac{1}{2}(x+16)}$
9. $-\dfrac{3}{9+x^2}$
10. $t^2e^{2t}(2t+3)$

CHAPTER 7

Exercise 7–1

1. $\dfrac{ax^2}{2}+bx+C$
2. $\dfrac{x^2}{2}+\dfrac{x^3}{3}+\dfrac{x^4}{4}+C$
3. $\dfrac{3x^{\frac{4}{3}}}{2}+C$
4. $8\sqrt{2x+1+C}$
5. $\dfrac{\left(x^4+\dfrac{x}{2}\right)^4}{8}+C$

Exercise 7–2

1. $\dfrac{K^{2t}}{2\, ln\, K}+C$
2. $\sin \dfrac{x^2}{2}+C$
3. $\dfrac{1}{2a}\, ln\,(av^2+b)\, C$
4. $\dfrac{\sin^4 3x}{12}+C$
5. $x^2-12x=4ln\,(x+2)+C$

Exercise 7–2—(Continued)

6. $e^{\tan x}+C$
7. $-\dfrac{1}{2}\cot (x^2-1)+C$
8. $e^{\sec x}+C$

Exercise 7–3

1. $\dfrac{1}{2}\, ln\,(\sec 2x+\tan 2x)+C$ (rule 16)
2. $\dfrac{1}{2}\sin^{-1}\dfrac{2t^2}{3}+C$ (rule 18)
3. $\dfrac{3}{4}\, ln\,(\sin 2\theta^2)+C$ (rule 15)
4. $\dfrac{\cos^3 2x}{6}-\dfrac{\cos^5 2x}{10}+C$ (rule 4)
5. $\dfrac{1}{3}\tan^{-1}\left(\dfrac{x-2}{3}\right)+C$ (rule 19)
6. $\dfrac{1}{9}\, ln \sec 3x^3+C$

   or $\dfrac{1}{9}\, ln \cos 3x^3+C$ (rule 14)
7. $-\dfrac{\cos 4x}{8}-\dfrac{\cos 3x}{6}+C$ (formula D)
8. $-\dfrac{\cos^8 2x}{16}+C$ (rule 4)
9. $\dfrac{x}{16}-\dfrac{\sin 12x}{192}-\dfrac{\sin^3 6x}{144}+C$ (formula A and B)
10. $\dfrac{1}{2}\text{ arc sin }\dfrac{x^2-3}{2}+C$ (rule 18)

CHAPTER 8

Exercise 8–1

1. $x \sin + \cos x + C$ (by parts)
2. $-\dfrac{\sqrt{4-x^2}}{4x}+C$ (by trigonometric substitution)
3. $ln\,\dfrac{t(t-2)}{(t+1)^2}+C$ (by partial fractions)
4. $\dfrac{e^{bx}(b \sin nx - n \cos nx)}{b^2+n^2}+C$ (by parts)

CHAPTER 8—(Continued)

Exercise 8-1—(Continued)

5. $\frac{2}{15}(x-4)^{\frac{3}{2}}(3x+8)+C$ (by algebraic substitution)

6. $\frac{-v}{9(v^2-9)^{\frac{1}{2}}}+C$ (by trigonometric substitution)

Exercise 8-2

1. 4 square units
2. $\frac{32\sqrt{3}}{9}$ square units
3. $\frac{9}{2}(\pi-2)$ square units

CHAPTER 9

Exercise 9-1

1. Convergent for $|x| \angle 2$
2. Convergent for $|x| \angle 1$
3. Convergent for $|x| \angle 1$

Exercise 9-2

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$
2. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots.$

Exercise 9-3

1. $ln\ x = ln\ a + \frac{1}{a}(x-a) - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3} \ldots.$
2. $\cos x = \cos a - (x-a)\sin a$

$$-\frac{(x-a)^2 \cos a}{2!}$$

$$+\frac{(x-a)^3 \sin a}{3!} + \ldots.$$

CHAPTER 10

Exercise 10-1

1. $x^m = Cy^n$ or $x^{\frac{1}{n}} = Cy^{\frac{1}{m}}$
2. $\sin y = C \cos x$
3. $y\ ln\ C(1-x) = 1$

Exercise 10-2

1. $2y = x^4 + Cx^2$
2. $y \sin x = x + C$
3. $9y = 3x - 1 + Ce^{-3x}$

Exercise 10-3

1. $y = C_1e^{3x} + C_2e^{-x}$
2. $y = C_1e^{-2x} + C_2xe^{-2x}$
3. $y = C_1e^{3x} + C_2e^{-3x}$

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